LOGICS FOR UNRANKED TREES

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Paper in the proceedings

- A much shortened version of the survey.
- Full version on my webpage
  - google.com $\Rightarrow$ `libkin` $\Rightarrow$ "I'm feeling lucky"
- Why? Limits on the number of pages in a single volume.

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- Why? Limits on the number of pages in a single volume.
- Apparently there is no Moore's Law in book binding technology:
  - Gutenberg's Bible, published in 1455 — 1282 pages
  - Springer's ICALP Proceedings, published in 2005 – 1477 pages

Trees are everywhere

- One of the most common objects we see in CS.
- Logics and automata on trees found many applications:
  - verification;
  - program analysis;
  - logic programming/constraint programming;
  - linguistics;
  - databases

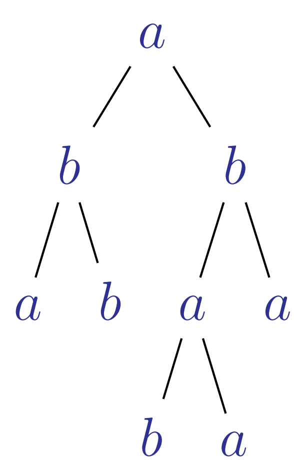
Logic/automata connection

- Regular tree languages = given by tree automata.
- Typically characterized via MSO — Monadic Second-Order logic
  - MSO is an extension of first-order logic with quantification over sets.
- Classical results (about 35 years old):
  - Thatcher-Wright: For finite binary trees, Regular = MS0-definable.
  - Rabin: Same is true for infinite trees. Hence S2S, the MSO-theory of two successors, is decidable.
  - This is one of the most powerful decidability results.
- Many more results followed (Thomas+colleagues, Wilke, Walukiewicz, Segoufin, Schwentick, Neven, etc etc)

Ranked Trees

Typically one works with ranked trees; e.g., binary, ternary, etc trees.

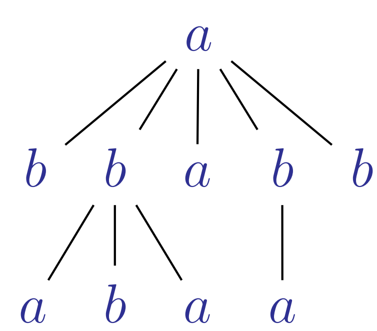
A binary tree:



Unranked trees

Recently focus has shifted towards unranked trees.

In them, nodes can have arbitrarily many children, and different nodes may have different number of children.



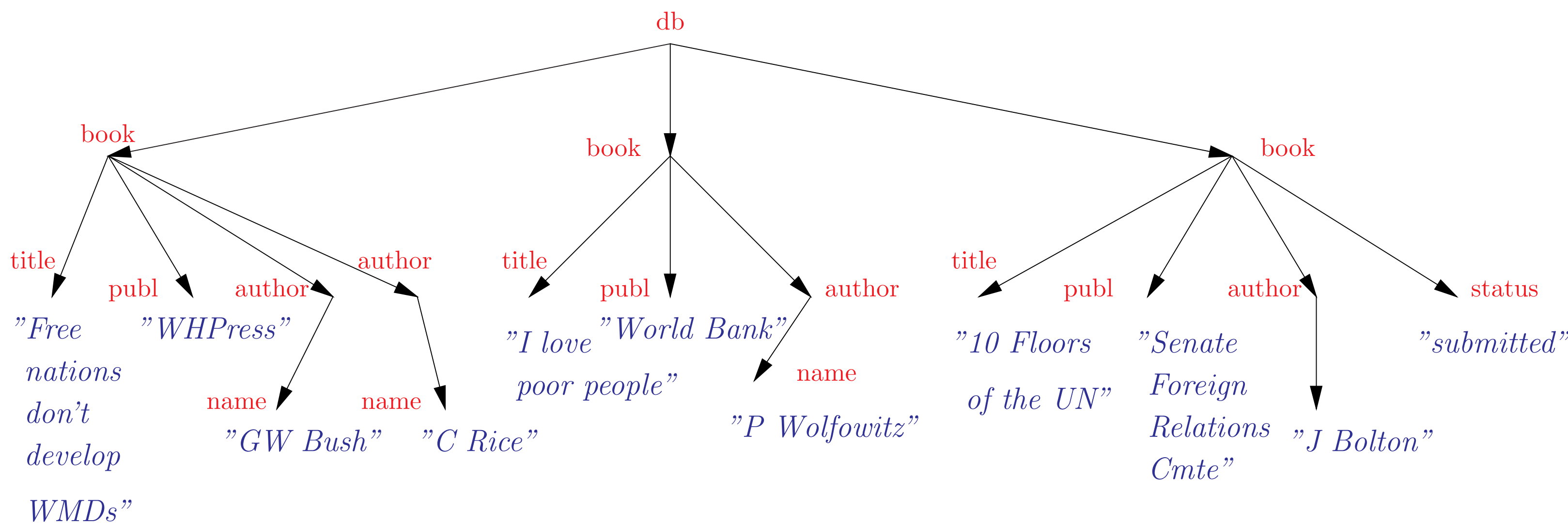
Why unranked trees?

- Main reason: XML.
- XML = eXtensible Markup Language, the standard for exchanging data on the web.
- XML data is modeled as unranked trees.
- A lot of recent work on XML: W3C standards such as XML Schema, XPath, XSLT, XQuery define types, navigation mechanism, transformations, and query languages for XML.
- Active work on XML in many communities, especially databases, information retrieval.
- Brings techniques (sometimes old and almost forgotten) from formal language theory and merges them nicely with logic.

XML documents look like this

```
<db>
  <book>
    <title attr_title="Free Nations don't develop WMD"></title>
    <publisher publ_attr="White House Press"></publisher>
    <author>
       <name name_attr="GW Bush">
    </author>
    <author>
       <name name_attr="C Rice">
    </author>
  </book>
  <book>
    <title attr_title="I Love Poor People"></title>
    <publisher publ_attr="World Bank Press"></publisher>
    <author>
       <name name_attr="P Wolfowitz">
    </author>
  </book>
  <book>
    <title attr_title="10 Floors of the United Naion"></title>
    <publisher publ_attr="Senate Foreign relations Committee"></publisher>
    <status status_attr="submitted"></status>
    <author>
       <name name_attr="J Bolton">
    </author>
  </book>
</db>
```

But we like to view them as unranked trees:



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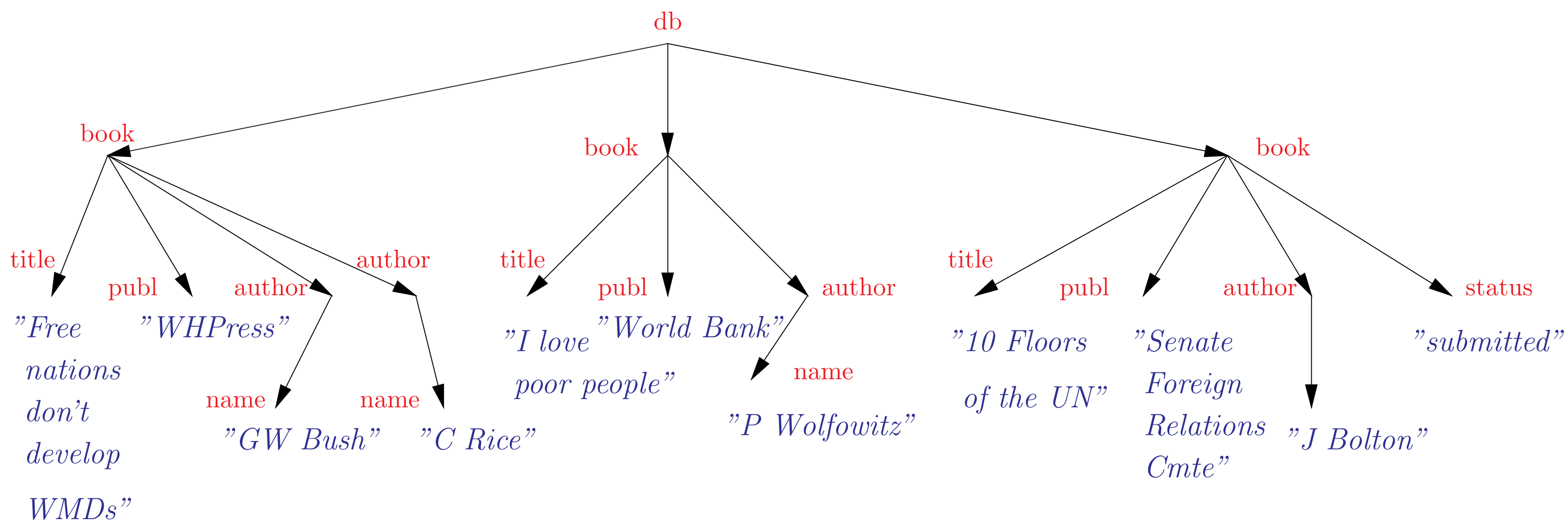


Document description (DTD = Document Type Definition)

$$\begin{aligned} \text{db} &\longrightarrow \text{book}^* \\ \text{book} &\longrightarrow \text{title}, \text{publ}, \text{author}^+, \text{status}? \\ \text{author} &\longrightarrow \text{name} \end{aligned}$$

plus attribute names.

Why are we interested in logics?

- XML documents describe data.
- Standard relational database approach:
  - data model – relations
  - declarative languages for specifying queries
  - procedural languages for evaluating queries
- Standard declarative languages are all logic-based:
  - relational calculus = first-order logic (FO)
  - datalog = fragment of fixed-point logic
  - basic SQL = FO with counting

What does XML add?

- New logics.
- New procedural languages:
  - logic–automata connection.

What do logics do?

Most commonly they define:

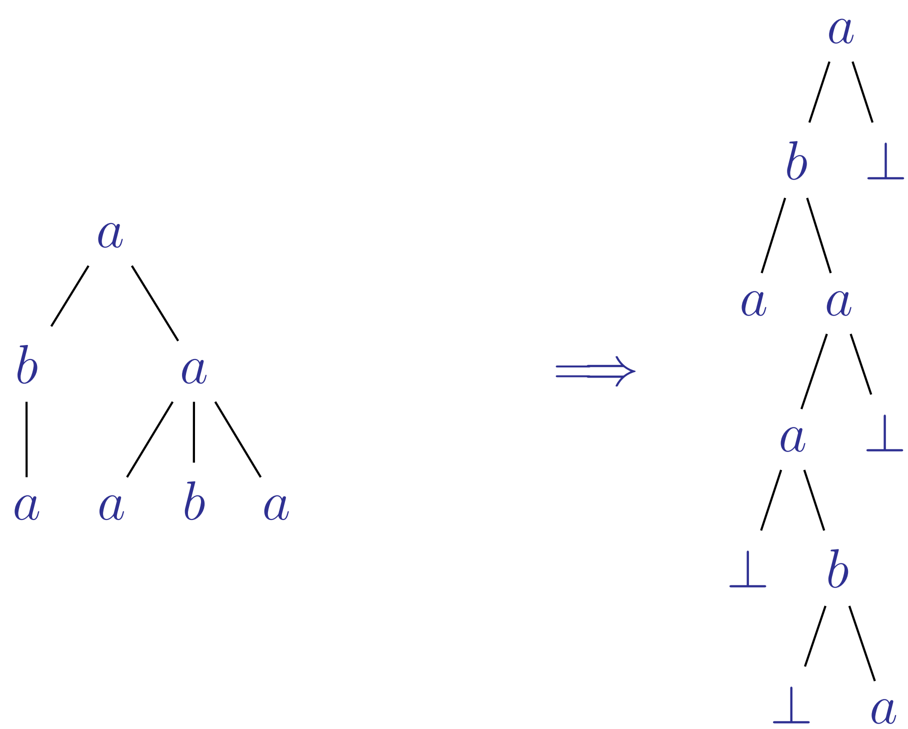
- Boolean (that is, yes/no) queries:
  - DTD conformance
  - Existence of certain paths
- Unary queries which select nodes in trees:
  - all nodes reachable by a certain path from the root;
  - all nodes with a certain label or certain data element.

Commonalities between logics

- (Almost) all have associated automata models.
- Crucial aspect is navigation.
- Hence we often see close connection with temporal logics.
- Logics are specifically designed for unranked trees.

Ranked/Unranked Connection

(used by Rabin in 1970 to interpret SωS in S2S):



It preserves recognizability by automata, MSO-definability, FO-definability...

Why not just use it?

- Instead of defining logics for unranked trees, just translate them into binary trees and use known logical formalisms.
- Problem: hard to navigate!
- A path in a translation becomes a union of arbitrarily many child paths and sibling paths.
- Hard (at least not natural) to express many properties such as DTD conformance.

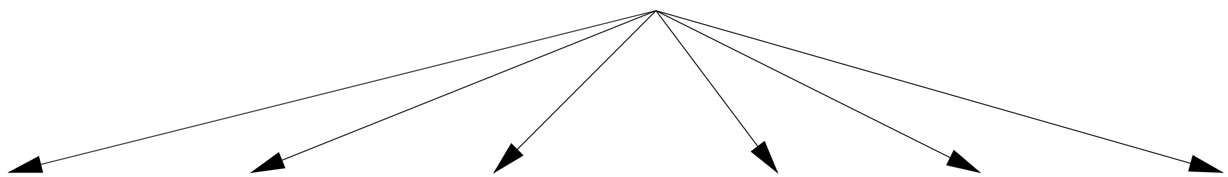
Classification: Yardstick logic

Most logics are based either on FO or MSO.

- FO:
  - Boolean connectives $\vee, \wedge, \neg$,
  - quantifiers $\exists x,\ \ \forall x$ ranging over nodes of trees.
- MSO: in addition
  - quantifiers $\exists X,\ \ \forall X$ ranging over sets of nodes;
  - plus new formulae $x \in X$.

Classification: Ordered vs Unordered Trees

In unordered trees, there is no order among siblings (children of the same node).



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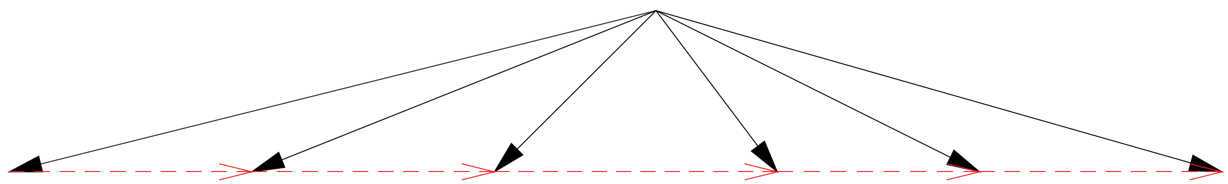
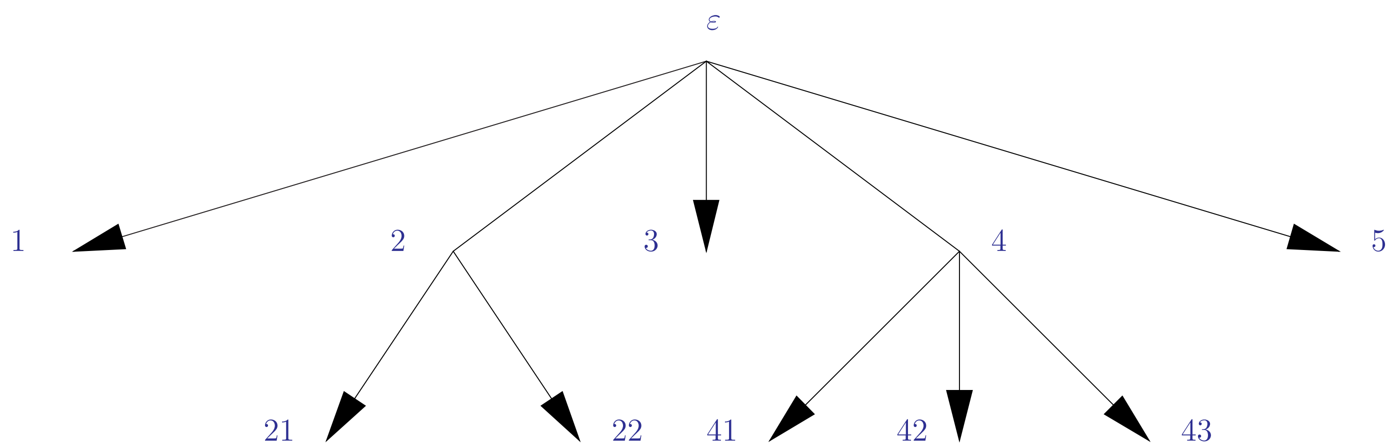


In ordered trees, siblings are ordered (from the oldest to the youngest).

Formal definition of unranked trees



Tree domain: prefix-closed subset D of $\mathbb{N}^*$ such that $s \cdot i \in D$ implies $s \cdot j \in D$ for $j < i$.

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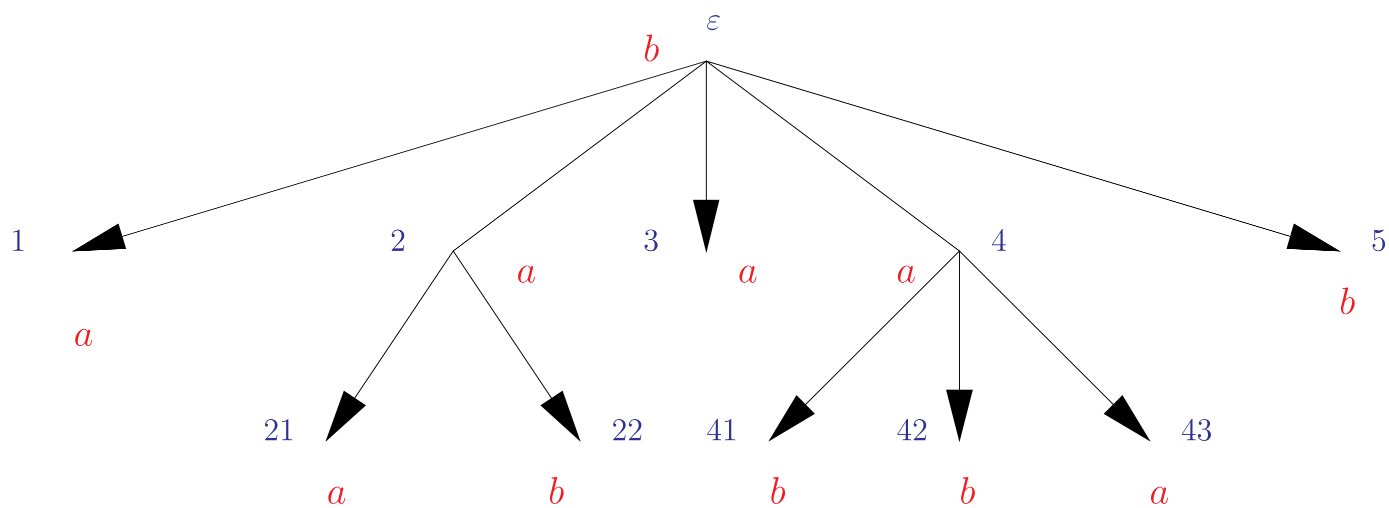

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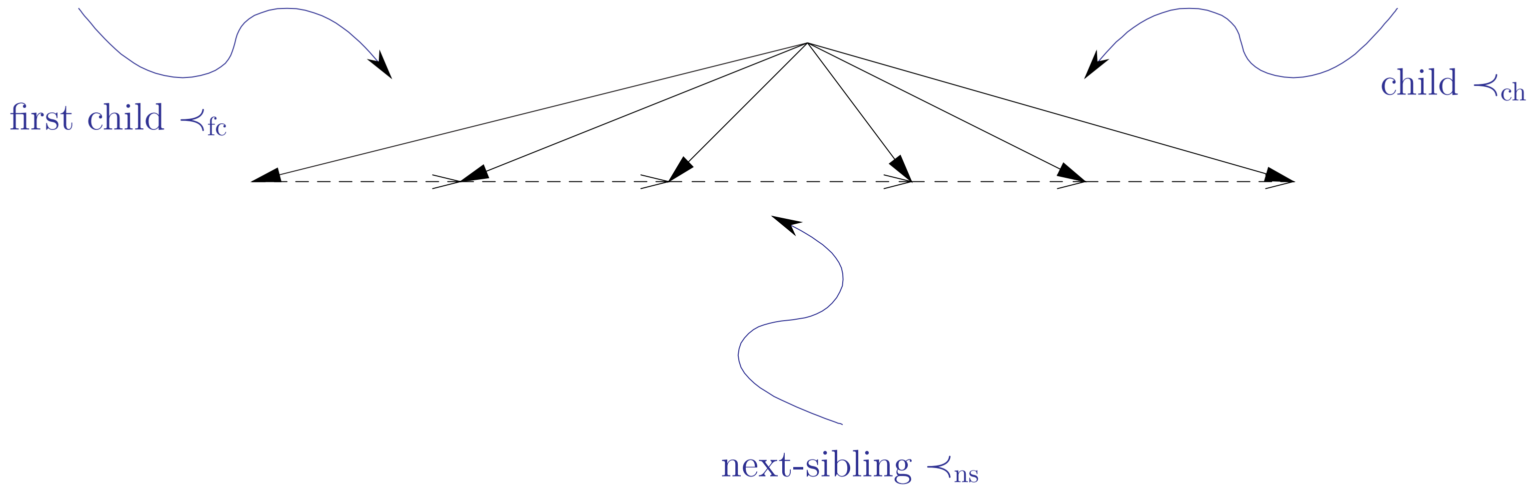
Tree over finite alphabet Σ: tree domain plus a mapping from it to Σ.

Basic predicates



Transitive closures:

- $\prec^*_{\mathrm{ch}}$ of $\prec_{\mathrm{ch}}$ (descendant)
- $\prec^*_{\mathrm{ns}}$ of $\prec_{\mathrm{ns}}$ (younger child)

We normally use transitive closures (since they are not definable in FO).

For MSO, we can use either $\prec_{\mathrm{ch}}, \prec_{\mathrm{ns}}$ or $\prec^*_{\mathrm{ch}}, \prec^*_{\mathrm{ns}}$ as they are interdefinable.

LOGICS FOR ORDERED TREES

Logic/automata connection

A set $\mathcal{T}$ of trees is definable in a logic $\mathcal{L}$ iff there is a sentence φ of $\mathcal{L}$ such that

$$T \in \mathcal{T} \quad \Leftrightarrow \quad T \models \varphi$$

A set $\mathcal{T}$ of trees is regular if it is recognizable by a tree automaton.

Theorem

- A set of binary trees is regular iff it is MSO-definable (Thatcher-Wright, 1966).
- A set of unranked trees is regular iff it is MSO-definable (almost folklore; stated many times by different authors).

Tree automata: the ranked case

Transitions are $\delta : \mathsf{States} \times \mathsf{States} \times \Sigma \to 2^{\mathsf{States}}$.

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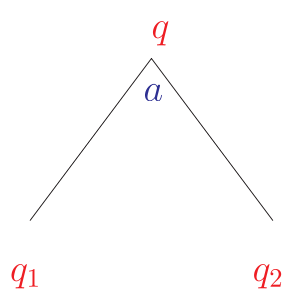

if $q \in \delta(q_1, q_2, a)$

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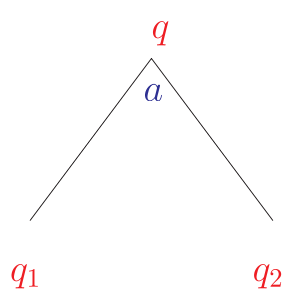

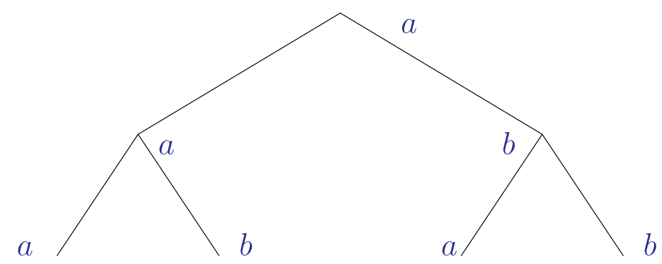

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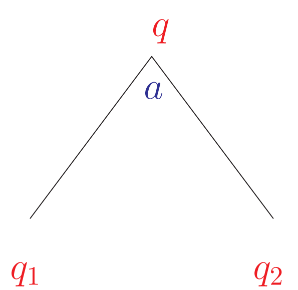

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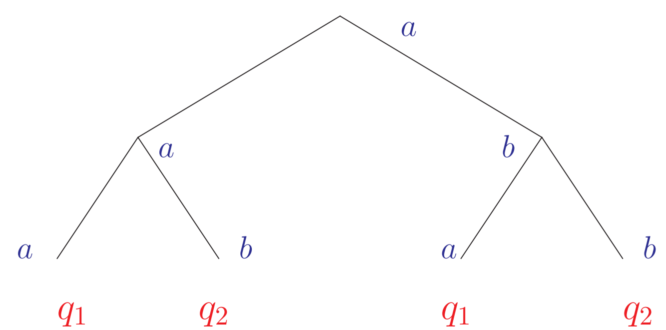

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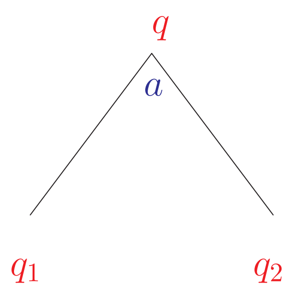

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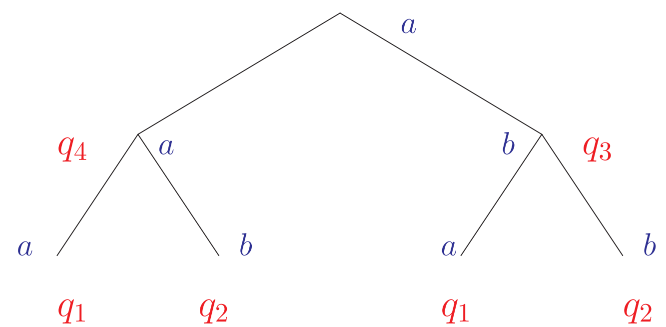

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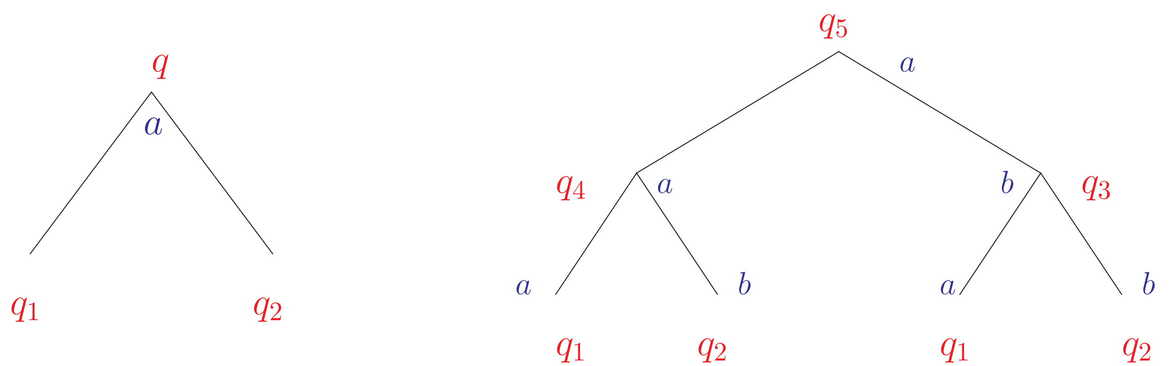

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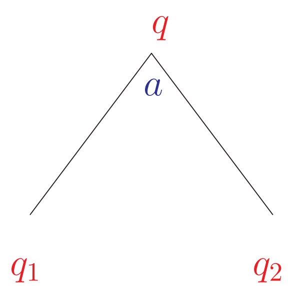

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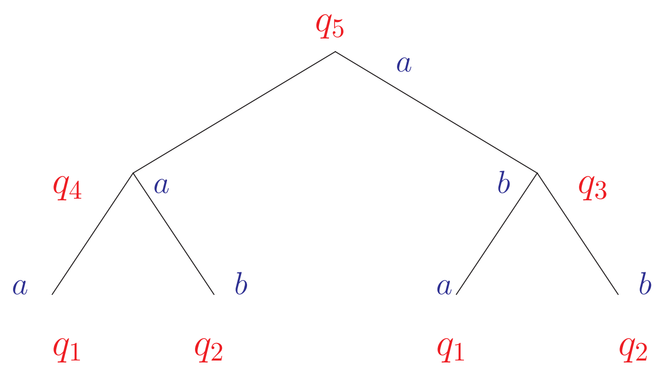

accepted if q_5 is a final state

Tree automata: unranked case

Transitions are $\delta : \mathsf{States} \times \Sigma \to 2^{\mathsf{States}^*}$ so that each $\delta(q, a) \subseteq \mathsf{States}^*$ is a regular language.

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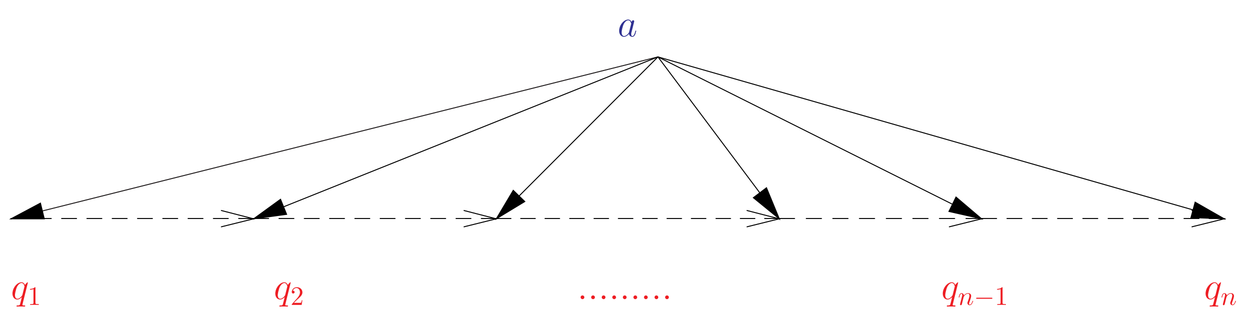

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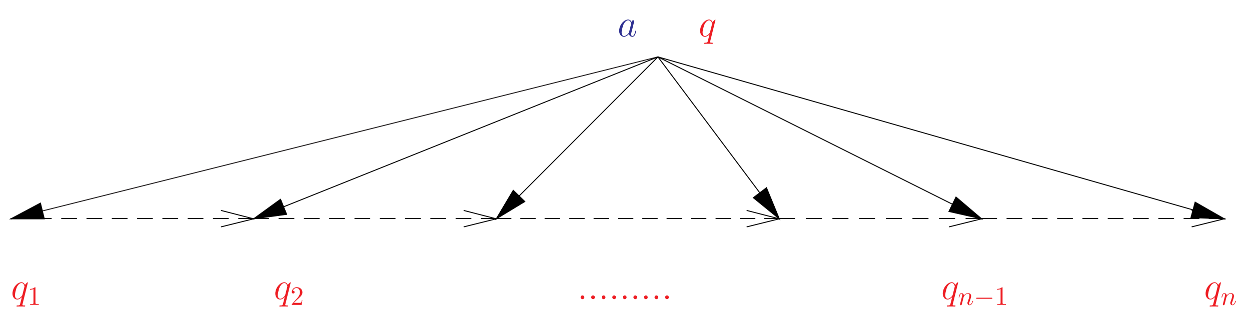

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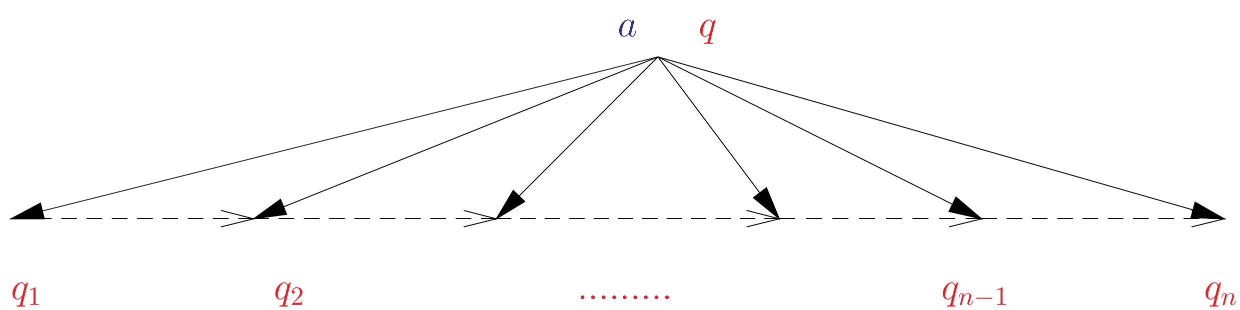

if $q_1 \cdots q_n \in \delta(q,a)$

A tree is accepted if there is a run such that the root is assigned an accepting state.

Unary queries

- A unary query selects a set of nodes in a tree.
- A surprisingly simple automaton model captures them.
- Query automaton QA $=$ unranked tree automaton $+$ selecting set $S \subseteq$ States $\times\ \Sigma$.
- QA selects a node s from a tree T if there is a run that assigns a state q to s such that
$$(q, a) \in S.$$

Theorem (Neven/Schwentick, 1999) For unary queries over unranked trees,

$$\text{Query Automata} \quad = \quad \text{MSO}.$$

MSO and DTDs

- Recall that DTDs have rules such as

$$\text{book} \quad \longrightarrow \quad \text{title}, \text{publ}, \text{author}^+, \text{status}?$$

- Since regular string languages are precisely those MSO-definable, it follows that all DTDs are MSO-definable.
- Are DTDs and MSO equal?
- The answer is negative.

MSO and DTDs cont'd

- EDTDs $=$ Extended DTDs: these are DTDs over a larger alphabet $\Sigma' \supseteq \Sigma$ together with a projection $\pi : \Sigma' \rightarrow \Sigma$.
- Trees over Σ that conform to an EDTD: projections of conforming trees over Σ'

Theorem (Thatcher 1967; rediscovered several times recently)

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- DTDs are not even closed under $\vee$ and $\neg$.
- Unions of DTDs correspond to the existential fragment of MSO over a smaller vocabulary.

MSO over trees: Complexity

- Model-checking problem:

INPUT:	tree T, sentence φ
QUESTION:	Is $T \models \varphi$?

- Two parameters:
  - $\|T\|$ – data complexity
  - $\|\varphi\|$ – query complexity
- By translation to automata: The data complexity of MSO is linear.
- Problem: if we keep data complexity linear, the query complexity is necessarily non-elementary! (Frick/Grohe, 2002)
- Can we do better?
- Yes, by finding different logics that have the power of MSO, and yet better model-checking properties.

Changing syntax to lower complexity: LTL

Syntax:

$$\varphi \;:=\; a(\in \Sigma) \;\mid\; \varphi \vee \varphi' \;\mid\; \neg\varphi \;\mid\; \mathbf{X}\varphi \;\mid\; \varphi\mathbf{U}\varphi'$$

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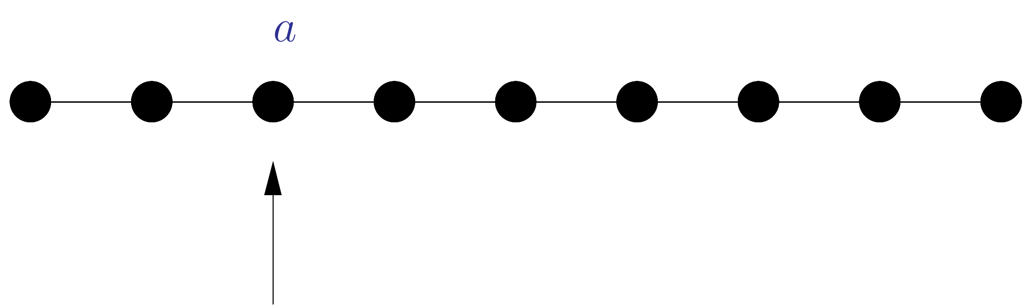

$$a, a \in \Sigma$$

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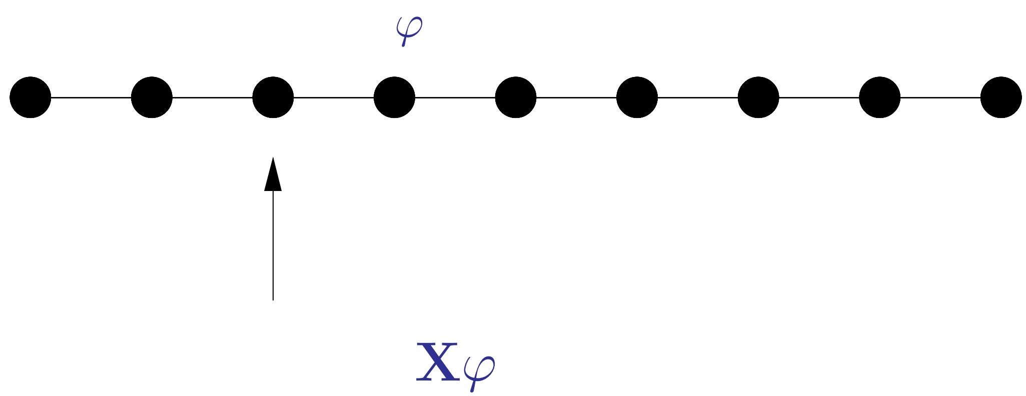

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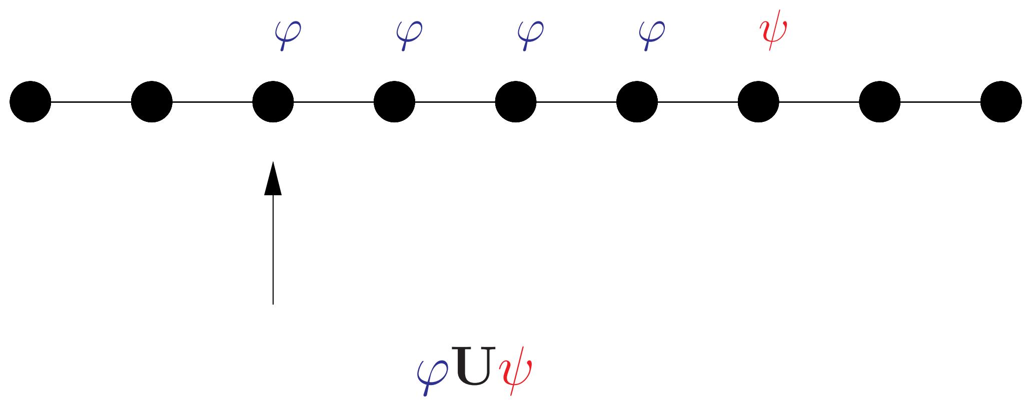


LTL cont'd

- LTL $=$ FO over strings (Kamp's theorem).
- To evaluate LTL with linear data complexity, one needs non-elementary query complexity
(modulo some complexity-theoretic assumptions; Frick/Grohe 2002)
- But LTL over strings can be evaluated in time

$$2^{O(\|\varphi\|)} \cdot |s|$$

- Of course this implies that translation from FO to LTL is non-elementary.

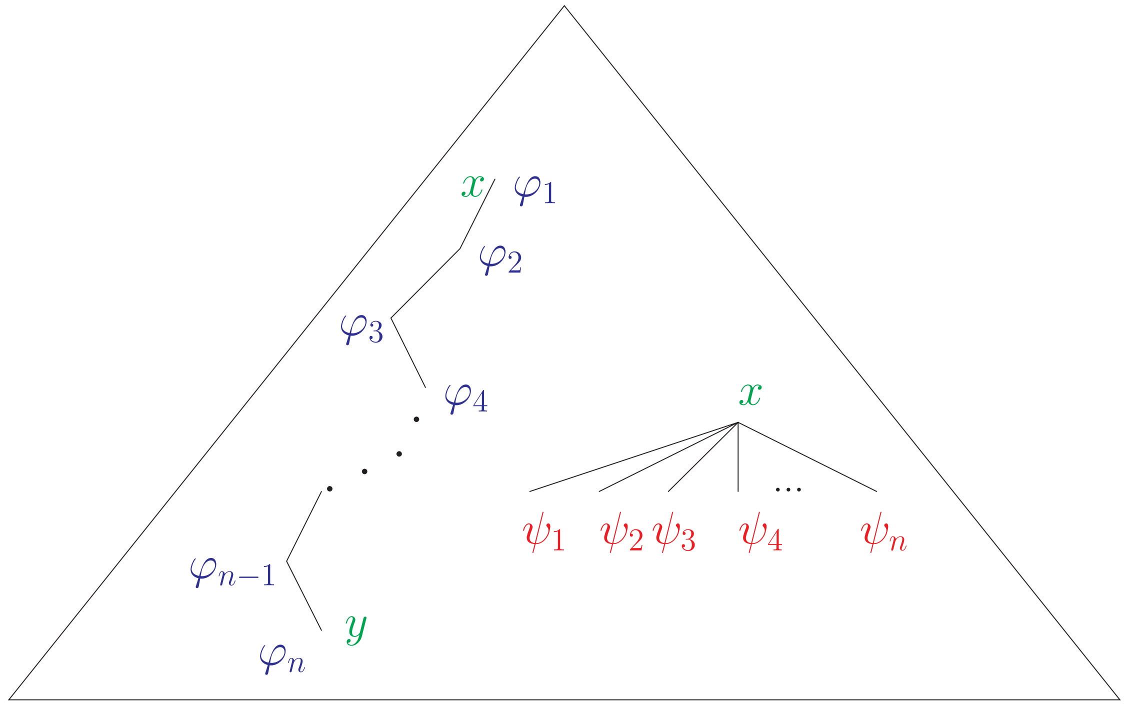
Efficient Tree Logic (ETL)

Neven/Schwentick, 2000

Idea of ETL: take MSO and

- Disallow:
  - next-sibling $\prec_{\mathrm{ns}}$;
  - arbitrary quantification;
- Add:
  - guarded quantification over children or (sets of) descendants;
  - regular expressions over formulae.
  - and put some syntactic restrictions.

Efficient Tree Logic (ETL) cont'd



$\varphi\_e(x, y): \quad \varphi_1 \cdots \varphi_n \in e$ $\qquad$ $\psi\_e(x): \quad \psi_1 \cdots \psi_n \in e$

Efficient Tree Logic (ETL) cont'd

Theorem (Neven/Schwentick)

- ETL $=$ MSO.
- ETL formulae can be evaluated in time

$$2^{O(\|\varphi\|)} \cdot \|T\|$$

Can one do better? – Monadic Datalog

- Datalog $=$ database query language; essentially extension of positive FO with least fixed point.
- Can also be viewed as prolog without function symbols.
- Datalog program is monadic if all introduced predicates (intensional predicates) are monadic – have one free variable.
- Example: select (in predicate D) all nodes s such that all their descendants (including s) are labeled a:

$$
\begin{array}{rcllll}
D(x) & :- & P_a(x), & \mathsf{Leaf}(x) & & \\
D(x) & :- & P_a(x), & x \prec_{\mathrm{fc}} y, & S(y) & \\
S(y) & :- & P_a(x), & \mathsf{LastChild}(y), & D(y) & \\
S(y) & :- & P_a(x), & x \prec_{\mathrm{ns}} y, & S(y), & D(y)
\end{array}
$$

Monadic Datalog cont'd

Assume that Leaf and LastChild are available as basic predicates.

Theorem (Gottlob/Koch, 2002)

- Monadic Datalog $=$ MSO
- A Monadic Datalog program $\mathcal{P}$ can be evaluated on a tree T in time

$$O(\|\mathcal{P}\| \cdot \|T\|)$$

μ-calculus over unranked trees

- μ-calculus (Kozen 82): extension of a temporal logic with the least fixed point operator.
- Subsumes many logics used in verification: LTL, CTL, CTL*.
- Syntax:

$$\varphi \ := \ S \mid a \mid \varphi \vee \varphi' \mid \varphi \wedge \varphi' \mid \neg\varphi \mid \mathbf{X}_E\varphi \mid \mu S\ \varphi(S)$$

  - S must occur positively;
  - E ranges over relations $\prec_{\text{ch}}$ and $\prec_{\text{ns}}$
- Full μ-calculus: one can talk about the past.
  - That is, E also ranges over inverses of $\prec_{\text{ch}}$ and $\prec_{\text{ns}}$

μ-calculus over unranked trees cont'd

Theorem (Barcelo, L., '05) Over unranked trees,

$$\text{full } \mu\text{-calculus} \quad = \quad \text{MSO}$$

For Boolean queries:

- MSO = alternation-free μ-calculus (all negations pushed to atomic formulae)
- Complexity of model-checking (Mateescu, '02):
  - $O(\|\varphi\|^2 \cdot \|T\|)$ for μ-calculus;
  - $O(\|\varphi\| \cdot \|T\|)$ for alternation-free μ-calculus.

Remark: it is well known that over infinite *binary* trees μ-calculus and MSO are the same (Niwinski 1988).

First-Order based formalisms

- These are often studied in connection with XPath
- XPath – a W3C standard, essentially the navigation language for XML.

XPath – an informal introduction

- XPath has two kinds of formulae: node tests and path formulae.
- Node tests are closed under Boolean connectives and can check if a path satisfying a path formula can start in a given node.
- Path formulae can:
  - test if a node test is true in the first node of a path;
  - test if a path starts by going to a child, first child, next child, previous child, parent, descendant, ansector, etc;
  - take union or composition of two paths.

Example: `//book[/author[name="GW Bush"]]/title`
gives titles of books coauthored by Bush.

CTL* vs XPath

- There is a well-known logic, CTL*, that similarly combines node (called *state*) and path formulae.
- Syntax:

state formulae	$\alpha \ := \ a \mid \alpha \vee \alpha' \mid \neg\alpha \mid \mathbf{E}\beta$
path formulae	$\beta \ := \ \mathsf{LTL}$ over state formulae

Example: all descendants of a given node (including self) are labeled a (with $\Sigma = \{a, b\}$):

$$\neg\mathbf{E}\left(\ (a \vee b)\ \mathbf{U}\ b\right)$$

CTL* and FO over trees

Theorem With respect to Boolean queries:

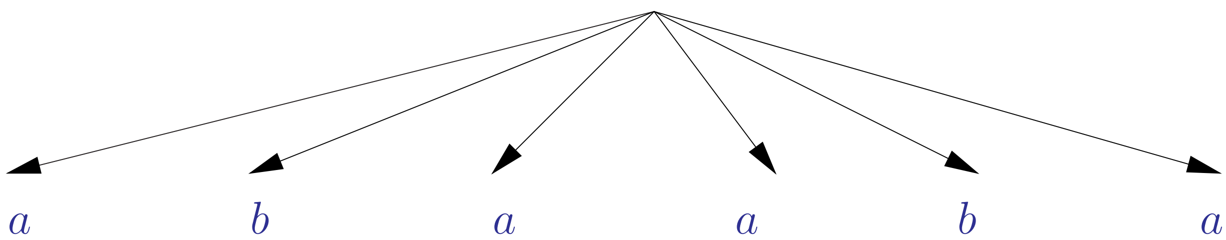
- over binary trees, CTL* $=$ FO (Hafer, Thomas, 1987).
- over unranked trees, CTL* $=$ FO (Barcelo, L., 2005; closely related to Marx 2004)
- For unary queries, one adds reasoning about the past (temporal operators $\mathbf{Y}$ – yesterday, and $\mathbf{S}$ – since).

- A technical issue: what is a path in an unranked tree? It could be a path that may change directions from siblings to children, or one could use two different kinds of path formulae.
- It turns out that this decision does not affect expressiveness.

LOGICS FOR UNORDERED TREES

Easy punchline

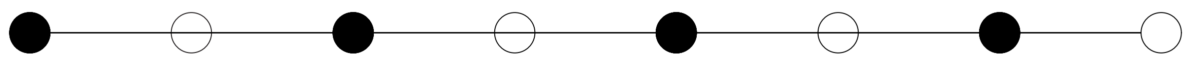
- Order buys us counting.
- Without order, counting has to be introduced explicitly.



- There is no way to say in a temporal logic that there are at least 2 children labeled a.

MSO, order, and counting

- With MSO, ordering gives us even more powerful modulo counting.
- Example: parity in MSO

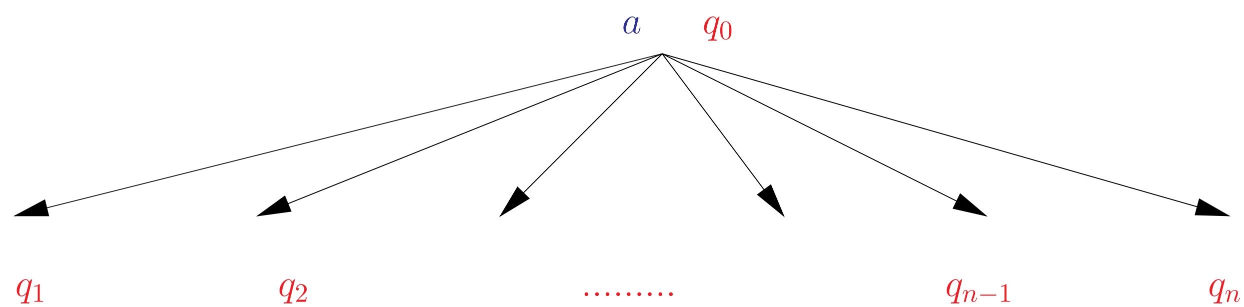


- The black set:
  - contains the first element;
  - contains every other element;
  - does not contain the last element.
- But if we only have:

we cannot say it.

Automata with counting

- New transition: δ : States $\times\ \Sigma \longrightarrow$ Boolean function over(V)
- $V \ = \ \{v_q^k \ \ | \ \ k \in \mathbb{N},\ q \in \text{States}\}$.
- A new notion of run:



- For each q, set v_q^k to true if the number of children in state q is at least k.
- If $\delta(q_0, a)$ evaluates to true, then state q_0 can be assigned.

Counting in temporal logics

Extend μ-calculus and CTL* to counting versions by changing $\mathbf{X}$ to $\mathbf{X}^k$, meaning the existence of at least k elements satisfying a formula.

Theorem For Boolean queries:

- MSO $=$ counting μ-calculus
  (Walukiewicz et al, 2002)
- FO $=$ counting CTL*
  (Moller, Rabinovich, 2003)
- For unary queries, one needs both counting and past
  (Schlingloff 1992, Barcelo, L, 2005)

Adding an arbitrary ordering

- Parity example: an order is needed, but it does not matter which one!
- Such properties are called order-invariant.

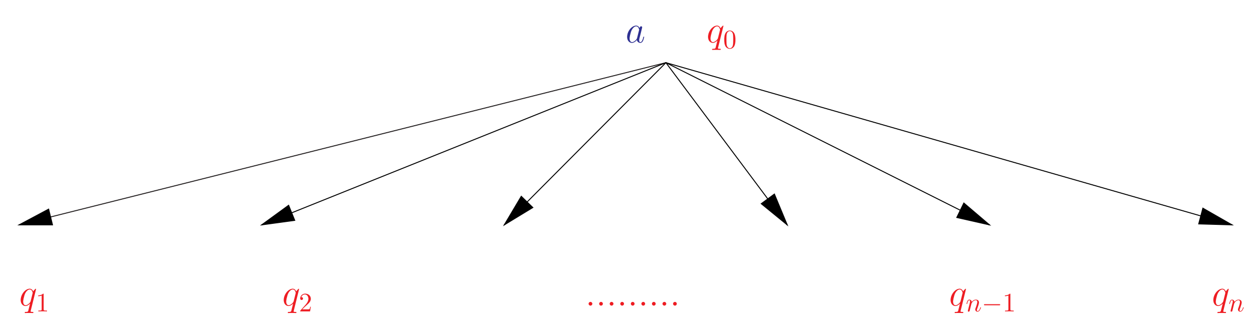
- CMSO = Counting MSO: extension of MSO with

$$\mathsf{Mod}_q(X) \qquad \text{meaning} \qquad |X| = 0 \ (\mathsf{mod}\ q)$$

Theorem (Courcelle 1991) Over trees,

$$\text{order-invariant MSO} \quad = \quad \text{CMSO}.$$

Even more powerful counting



- For each q, let v_q be the number of nodes assigned state q.
- In a more powerful counting automata, transition $\delta(q_0, a)$ could be a formula of Presburger Arithmetic (that is, $\langle \mathbb{N}, + \rangle$) over v_q's.
  For example,

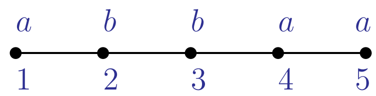
$$\delta(q_0, a) \;=\; \left(v_{q_1} + v_{q_2} = 2 \cdot v_{q_3}\right) \wedge \exists x \; \left(v_{q_4} = x + x + x\right)$$

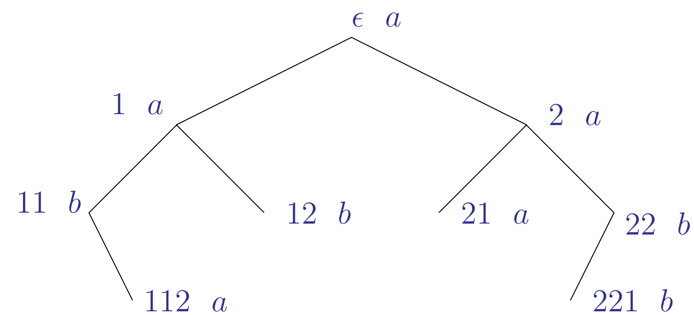
- Such automata investigated by Seidl, Schwentick, Muscholl, '03–04.
- Decidability results and fixed-point logic characterizations.

AUTOMATIC STRUCTURES

Strings, trees, and logic: a reminder

Strings and trees are viewed as finite structures.
Universe: $\{1, \ldots, n\}$ or a prefix-closed subset of $\{1, 2\}^*$
Predicates: $\prec$ – prefix, P_a and P_b for positions labeled a and b





A string s or a tree T is a structure, M_s or M_T, of vocabulary $(\prec, P_a, P_b)$.
If Φ is a sentence of a logic $\mathcal{L}$, then

$$\{s \mid M_s \models \Phi\} \qquad \{T \mid M_T \models \Phi\}$$

define string and tree languages.

Classical Results Regular languages $=$ MSO-definable

A different approach: automatic structures

If $\mathcal{M} = \langle \Sigma^*, \Omega \rangle$ is a first-order structure, then a formula $\varphi(x)$ defines the language

$$\{s \in \Sigma^* \mid \mathcal{M} \models \varphi(s)\}$$

A structure is automatic if all such languages are regular.

Operations on strings:	• $x \prec y$: x is a prefix of y • $f_a(x) = x \cdot a$, $a \in \Sigma$ • $\mathsf{el}(x, y)$ (equal length): $|x| = |y|$

$$\boxed{\mathfrak{S} \quad \stackrel{def}{=} \quad \langle\ \Sigma^*,\ \prec,\ (f_a)_{a\in\Sigma},\ \mathsf{el}\ \rangle}$$

Folklore Theorem $\mathfrak{S}$ is the universal automatic structure: relations definable by formulae $\varphi(x_1, \ldots, x_n)$ are precisely the regular relations.

Regular relations

These are n-tuples of strings accepted by letter-to-letter automata.

$$
\begin{array}{rclllllll}
s_1 & = & \mathsf{a} & \mathsf{a} & \mathsf{b} & \cdots & \mathsf{a} & \mathsf{b} & \mathsf{c} \\
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A reduct of $\mathfrak{S}$: $\boxed{\mathfrak{S}_{\mathfrak{p}} \stackrel{def}{=} \langle\ \Sigma^*,\ \prec,\ (f_a)_{a\in\Sigma}\ \rangle}$

Theorem (Benedikt, L., Schwentick, Segoufin, 2001)
Languages definable over $\mathfrak{S}_{\mathfrak{p}}$ are precisely the star-free languages.

Automatic structures on binary trees

$\mathsf{Trees}(\Sigma)$ – the infinite set of all binary Σ-labeled trees.

Operations and predicates on trees:

T' extends T
(or T is subsumed by T'): $T \prec T'$

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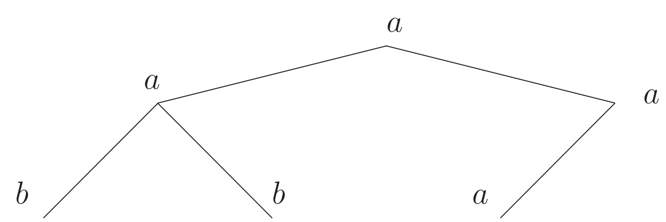

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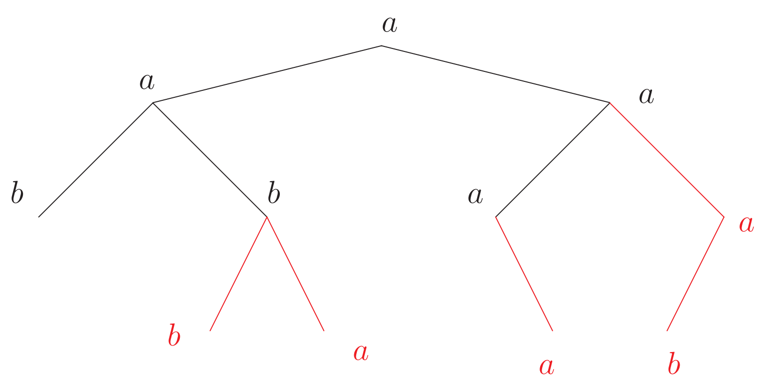

Operations on binary trees cont'd

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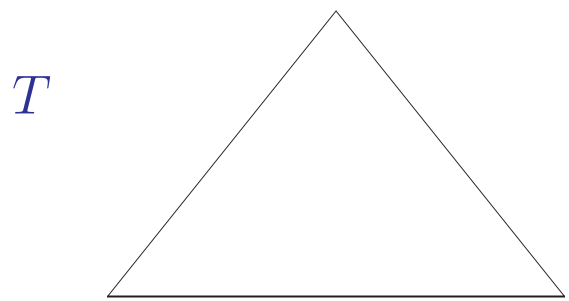

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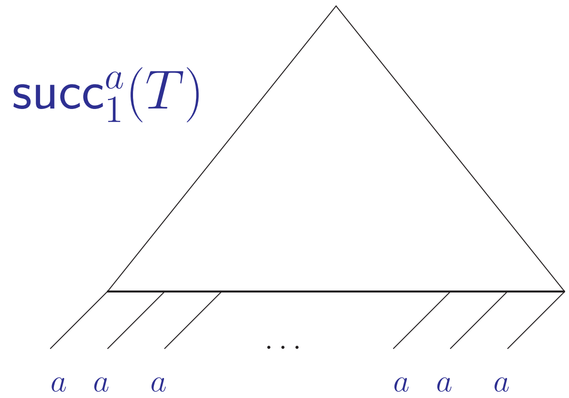

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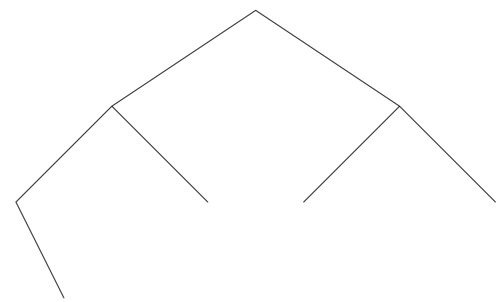

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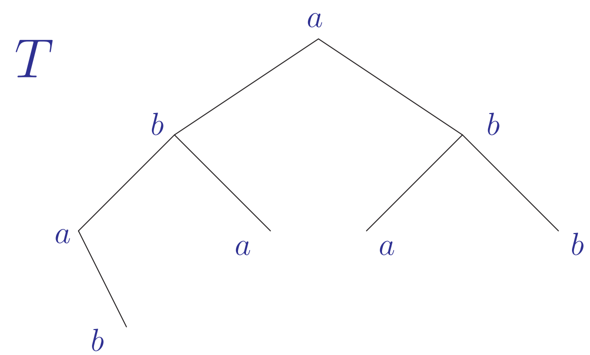

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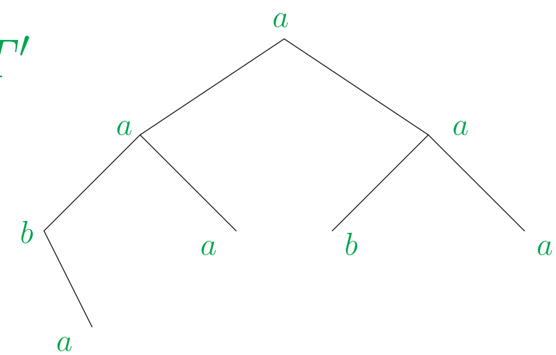

Tree-Automatic Structures

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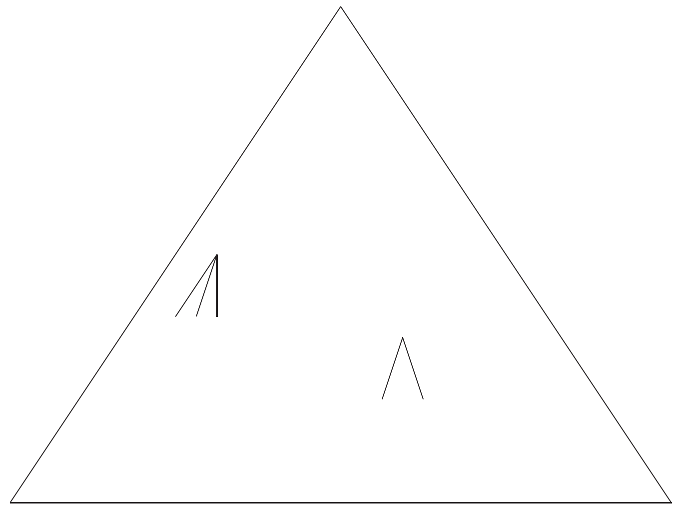
Theorem (Benedikt, L., 2002)

- For both $\mathfrak{T}_{\mathfrak{p}}$ and $\mathfrak{T}$, the class of definable sets is precisely the class of regular tree languages.
- $\mathfrak{T}$ is the universal tree-automatic structure: a relation on $\mathsf{Trees}(\Sigma)$ is $\mathfrak{T}$-definable iff it is regular.
- $\mathfrak{T}_{\mathfrak{p}}$ is weaker than $\mathfrak{T}$.

Operations on unranked trees

Reusing the extension operation $\prec$ requires infinitely many successor operations, which is undesirable. Hence, we split it into two: extension right $\prec_{\rightarrow}$ and extension down $\prec_{\downarrow}$.



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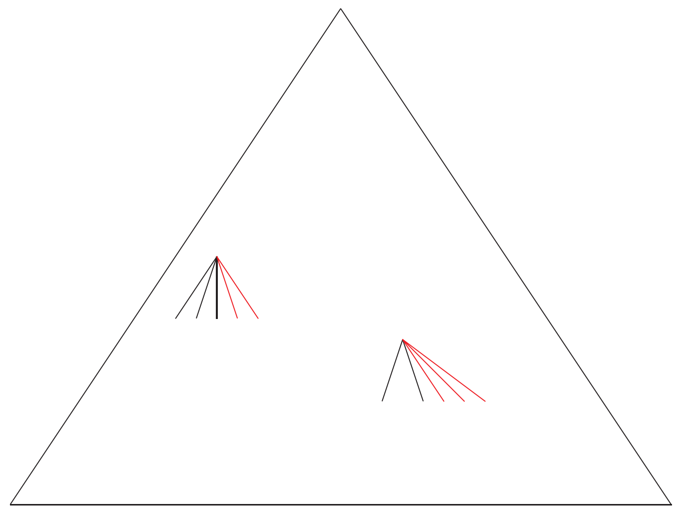

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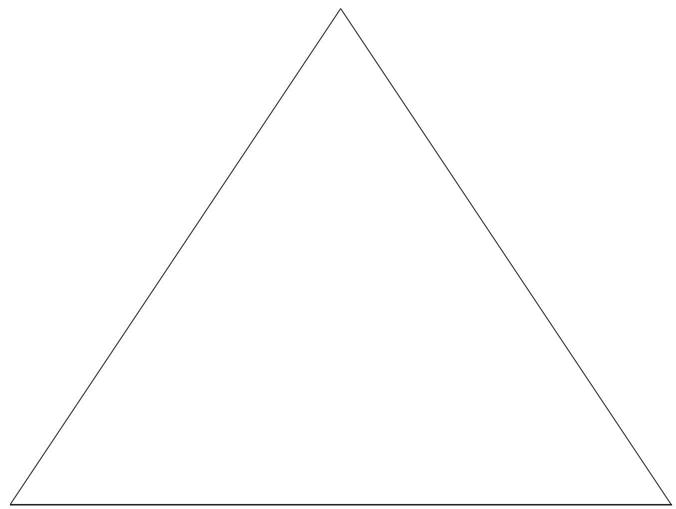

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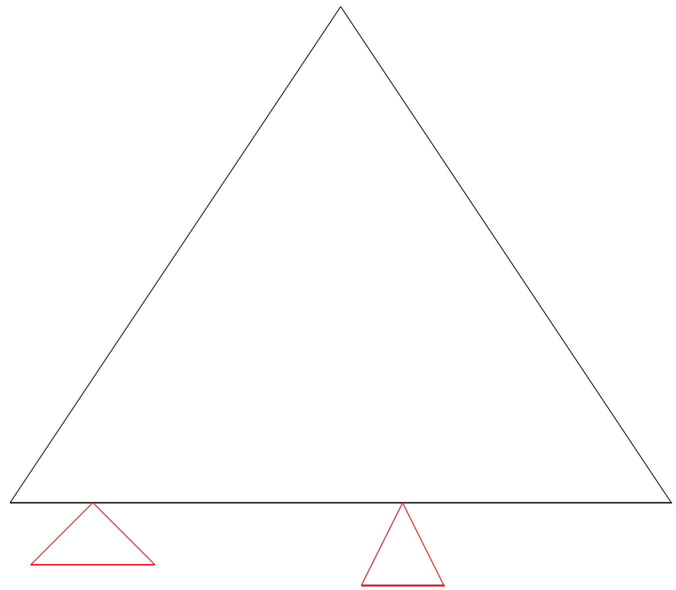

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Unranked Tree-Automatic Structures

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Unranked Tree-Automatic Structures: basic results

Theorem (L., Neven, 2003)

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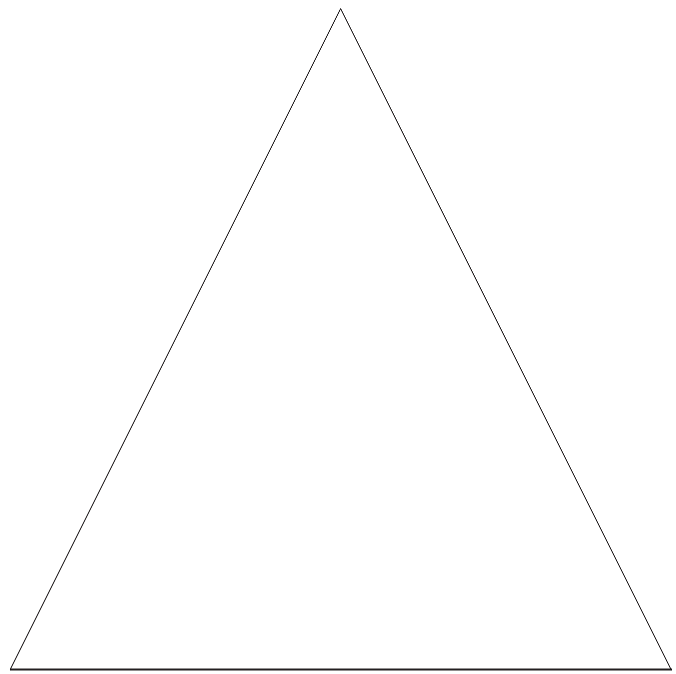

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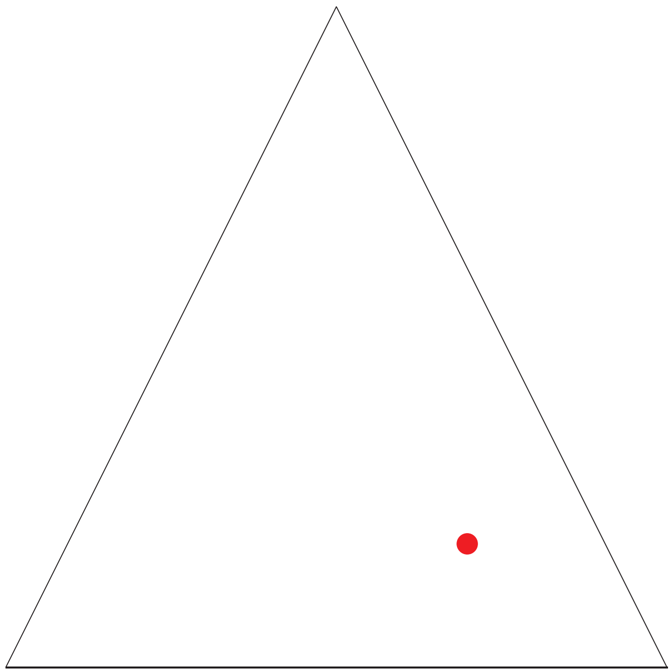

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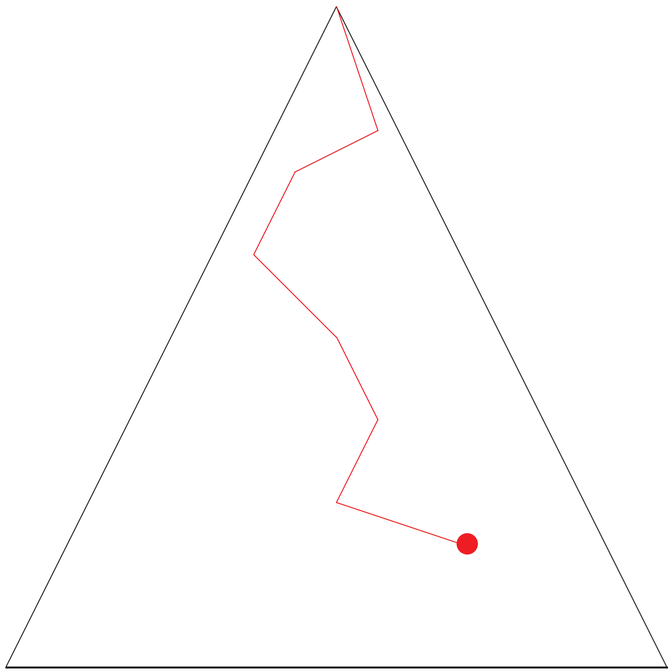

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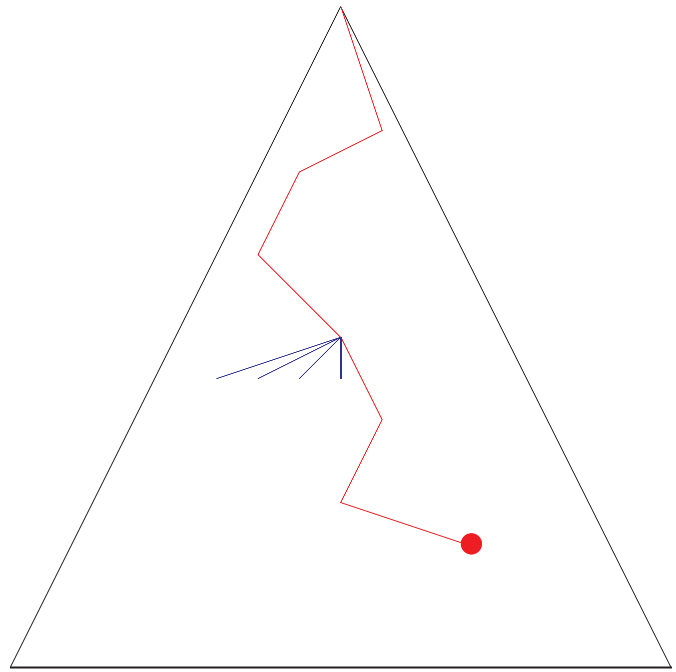

Unranked branch

Logics with branch quantification

We write **FO**$_\eta$ to indicate that we quantify only over branches.

Then definable sets of trees have analog in the classical theory of logical definability over trees, which uses logics such as $\mathcal{FO}$, $\mathcal{MSO}$, $\mathcal{MSO}^{path}$ (only quantification over chains is allowed).

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Theorem Over ranked trees:

$$\mathbf{FO}_\eta(\mathfrak{T}_\mathfrak{p})\text{-definable} = \mathcal{FO}\text{-definable}$$

$$\mathbf{FO}_\eta(\mathfrak{T})\text{-definable} = \mathcal{MSO}^{path}\text{-definable}$$

Logics with branch quantification

We write $\mathbf{FO}_\eta$ to indicate that we quantify only over branches.

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Over unranked trees:

$$\mathbf{FO}_\eta(\mathfrak{T}_\mathfrak{p}^\mathfrak{u})\text{-definable} = \mathcal{FO}\text{-definable}$$
$$\mathbf{FO}_\eta(\mathfrak{T}^\mathfrak{u})\text{-definable} = \mathcal{MSO}^{\uparrow}_{\rightarrow}\text{-definable}$$

$\mathcal{MSO}^{\uparrow}_{\rightarrow}$ is $\mathcal{MSO}$ with quantification restricted to vertical and horizontal paths: an analog of $\mathcal{MSO}^{path}$ for unranked trees.

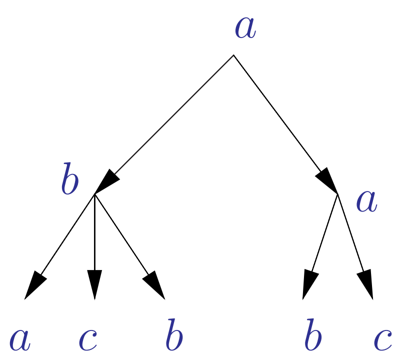
What else is in the survey?

- Edge-labeled trees.
- They occur in a variety of areas:
  - computational linguisticts;
  - ambient and spatial logics.
- Logics have quite a different flavor.
- Connections between them and logics considered here are being explored.

Other directions

- We have seen plenty of declarative specification languages with good associated procedural formalisms in terms of model-checking properties.
- What causes them to be good?
- One way to look at this: succinctness (Grohe/Schweikardt).
  How big are formulae for expressing certain properties?

Other directions: streaming

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Streamed representation:

$$aba\bar{a}c\bar{c}b\bar{b}\bar{b}ab\bar{b}c\bar{c}\bar{a}\bar{a}$$

Question: what properties of trees can we check by a finite automaton over the streamed representation?

Since the language of balanced parentheses is not regular, we may assume the input is already a valid stream.

Other directions: streaming cont'd

- Example The following DTD is not stream-verifiable (Segoufin/Vianu 2002):
  $a \rightarrow ab \mid ca \mid \varepsilon$
  $b \rightarrow \varepsilon$
  $c \rightarrow \varepsilon$
- Originally an involved pumping-lemma argument, but logic gives a much simpler proof:
- For every MSO sentence φ one can find two strings of the form

$$ab\bar{b}ab\bar{b}\ldots ab\bar{b}a\ldots a\bar{a}c\bar{c}\ldots\bar{a}c\bar{c}\bar{a}\ldots\bar{a}$$

  that agree on φ; one of them corresponds to the above DTD, and the other one to:
  $a \rightarrow a \mid ab \mid ca \mid \varepsilon$
  $b \rightarrow \varepsilon$
  $c \rightarrow \varepsilon$

Other directions: streaming cont'd

- There is a characterization of a fragment of MSO over trees that defines precisely the "streamable" properties (checked by string automata).
- However, decidability of that fragment remains open.

Other directions: data values

- So far we considered only labels on trees (e.g., book, author) but no data values (e.g., "WH Press").
- Adding data values quickly leads to undecidability.
- Example: DTDs + key/foreign key constraints.
  - Satisfiability problem: is a specification consistent?
  - Some known results (Fan, L., 2001):
  - It is NP-complete for unary constraints (e.g. title determines publisher).
  - It is undecidable even for binary constraints (e.g., title and author determine publisher).

Other directions: data values

- Proofs were not logic-based (mostly combinatorial plus integer linear programming).
- But it appears that logic can provide an explanation.
- Consider strings with data values attached to positions.
- Bojanczyk, Muscholl, Schwentick, Segoufin, 2005:
  - FO^2, first-order with two variables, is decidable.
  - FO^3, first-order with three variables, is undecidable.
- One needs two variables to talk about unary constraints, and more for binary, etc., constraints.

Summing up

- XML application give theoreticians nice problems to work on.
- Combination of well developed tools:
  - formal languages,
  - logic,
  - string and tree automata
- Not everything is a straightforward adaption of old and known results.