Winter 2003 CSC2531 CONSTRAINT DATABASES

Homework

assigned continuously, due likewise

Choose the problems you want to work on. If a problem has several items, at least 67% of them must be done. You need to solve roughly 50% of the problems if you're enrolled in the course.

Problem 1 Suppose you are given a relation $S(x, y, t)$ in $\mathbb{R}^3$ and a relation $S'(x, y)$ in $\mathbb{R}^2$. Show that the following can be expressed in $\mathrm{FO}(\mathbf{R}, \{S, S'\})$:

- S is a graph of function $f_S : \mathbb{R}^2 \to \mathbb{R}$ (or S is a graph of function $f_{S'} : \mathbb{R} \to \mathbb{R}$.
- The functions f_S and $f_{S'}$ are continuous.
- $f_{S'}$ is C^k, for a given $k > 0$.
- Assume that S defines a trajectory of an object: position (x, y) at time t. Assume that S has all the right properties. Define a formula $\varphi(x, y, v)$ that finds the speed of the object at (x, y).
- Assume that $S \subseteq \mathbb{R}^2$ is a convex polyhedron. Define its triangulation, that is, a formula $\varphi(\vec{x}^1, \vec{x}^2, \vec{x}^3)$ which is true iff $\vec{x}^i = (x^i, y^i)$ are the vertices of one of the triangles in the triangulation.
- Does the construction extend to $\mathbb{R}^3$? Why?
- Show that $\mathrm{FO} + \textsc{Lin} = \mathrm{FO}(\mathbf{R}_{\mathrm{lin}}, \sigma)$ can test if a piece-wise linear function is continuous.

Problem 2 Let $\mathcal{M} = \langle \mathbb{R}, +, \cdot, e^x \rangle$. Let $S \subseteq \mathbb{R}^2$ be definable over $\mathcal{M}$.

- Prove or disprove: $\mathrm{FO}(\mathcal{M}, \{S\})$ can test if S is a graph of a polynomial.
- Same question under the assumption that S is semi-algebraic (definable over $\mathbf{R}$).

Problem 3 Prove the total active-generic collapse ($\mathrm{FO}_{\mathrm{act}}(\mathcal{M}, \sigma) = \mathrm{FO}_{\mathrm{act}}(\sigma)$, as opposed to $\mathrm{FO}_{\mathrm{act}}(\mathcal{M}, \sigma) = \mathrm{FO}_{\mathrm{act}}(\sigma, <)$) for the following $\mathcal{M}$:

- $\langle \mathbb{R}, +, \cdot \rangle$;
- $\langle \mathbb{N}, \mathit{successor} \rangle$
- $\langle \mathbb{C}, +, \cdot \rangle$

Problem 4 Prove that no locally-generic query extends the active domain of its input (definition of locally-generic Q is in the text; it means that a query returns a finite output and for any monotone partial function $\pi : U \to U$ such that π is defined on $adom(\mathfrak{A}) \cup adom(Q(\mathfrak{A}))$, we have $Q(\pi(\mathfrak{A})) = \pi(Q(\mathfrak{A}))$).

Problem 5 Prove the natural-active collapse for the structure

$$\langle \mathbb{Q}, < \rangle$$

and analyze its complexity in two settings:

- the complexity of the natural-active collapse, that is, when quantifier-elimination algorithm is applied;
- the complexity of the RQC, that is, when $\mathrm{FO}(\langle \mathbb{Q}, < \rangle)$ are still allowed.

Problem 6 Same as in Problem 5 for

$$\langle \mathbb{Z}, +, -, 0, 1, <, (\sim_k)_{k>1} \rangle$$

where $\sim_k$ are binary relations $x = y (\bmod\ k)$.

Hints for 5 and 6: the complexity in 6 should be lower than for the real field, the complexity in 5 should be *significantly* lower.

Problem 7 This is an easy one:

1. Find an outrageous bug/typo in the proof of Theorem 5.2.2 in the text.
2. Fix it.

Problem 8 Give a direct proof (without using local triviality) that adding the connectivity operator $\mathbf{C}$:

$$\mathfrak{A} \models \mathbf{C}\vec{y}\ \varphi(\vec{y}, \vec{a}) \quad \text{iff} \quad \{\vec{b} \mid \mathfrak{A} \models \varphi(\vec{b}, \vec{a})\} \text{ is connected}$$

to $\mathrm{FO}(\mathbf{R}_{\mathrm{lin}}, \sigma)$ gives us a closed language (that is, each query returns a semilinear output on a semilinear input).

Problem 9 Prove that in monadic existential second-order logic over $\mathbf{R}$ and σ, one can express every computable property of finite σ-structures. How many second-order quantifiers does one need?

Problem 10 Consider $\mathcal{M} = \langle \Sigma^*, (f_a)_{a\in\Sigma}, (l_a)_{a\in\Sigma} \rangle$, where $f_a(x) = a \cdot x, l_a(x) = x \cdot a$.

Prove that $\mathcal{M}$ has the restricted quantifier collapse. You are not allowed to use the result that the extension of $\mathcal{M}$ with the prefix relation has the RQC.