Assignment 1

Exercises Choose any 5 from those below.

From the text: 3.5(2), 3.5(5), 3.7, 3.9, 3.12, 3.13, 3.17, 4.15, 4.12, 4.16, 5.1, 6.1, 6.2.

and

Consider two countable undirected graphs. For G_1, the universe is $\mathbb{N}$, and we have an edge between i and $j > i$ iff $\mathrm{BIT}(i, j)$ is true. For G_2, the universe is $\mathbb{N} - \{0\}$, and there is an edge between i and $j > i$ iff j is divisible by the ith prime. Prove that G_1 and G_2 are isomorphic.

Due date: November 15.

Problems. These cover the entire course; Assignment 2 will only have exercises. The due date is the mid or late December.

Choose one or two you want to work on. It's ok to work in groups. It's better to have a complete solution for one problem rather than make progress towards solving two.

(1) Prove that $+$ and $\times$ are expressible in $\mathrm{FO}(<, \mathrm{BIT})$.

(2) Show how to express $|X| \leq \log_2 |Y|$ in $\mathrm{FO}(<, \mathrm{BIT})$.

(3) Prove Gaifman's theorem, and establish the bound $O(4^{\mathsf{qr}(\varphi)})$ on r (rather than $7^{\mathsf{qr}(\varphi)}$ given in the text).

(4) Prove that MSO does not have the zero-one law.

(5) Prove that FO on planar graphs is fixed-parameter tractable.

(6) Prove that graph connectivity is not definable in $\exists$MSO, even over ordered graphs.

(7) Prove that the arity hierarchy of SO is strict: for each n, there is a SO-definable property which is not definable with quantification over relations of arity at most n.

(8) Provide logical characterization of EXPTIME (or NEXPTIME, choose one).

(9) Same for the class of problems of elementary data complexity.

(10) Prove that there is a property of ordered structures definable in FO^3 but not in FO^2.

Challenges.

(α) Use Ehrenfeucht-Fraïssé games to prove that parity is not definable in $\mathrm{FO}(\mathsf{All})$.

(β) Prove that every invariant $\mathrm{FO}(<, +)$ query is Gaifman-local (you may find it convenient to start with the case of one free variable, and prove weak locality first).

(γ) Prove that $(\mathrm{FO} + <)_{\mathrm{inv}}$ queries are Hanf-local.

(δ) Prove that for every $k > 2$, there is a property of ordered structures definable in FO^{k+1} but not in FO^k. If this is hard, try $k = 3$ first.