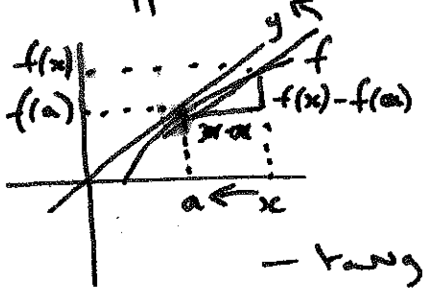
Chapter 01: Functions and models

1. Differencial & integral calculus



$y = mx + b$

$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

— tangent pb —

dif | int

$x - a = d$

$A_{f,d} = \lim_{n \to \infty} A_n$

— area pb —

2. Vocabulary

- A function f is defined on a domain D
- f(x) is the value of f at x
- The range of f is the set of all possible values of f(x) a x varies
- x is an independent variable, f(x) is a dependent variable

→ $\forall x \in D_f, f(x) = f(-x) \Rightarrow f$ is an <u>even</u> func°

→ $\forall x \in D_f, -f(x) = f(-x) \Rightarrow f$ is an <u>odd</u> func°

→ Vertical line test: a curve in the xy-plane is the graph of a function of x if and only if there is no vertical line that intersects the curve more than once.

→ Absolute value function is a piecewise defined function

→ Linear function: $f(x) = \underbrace{a}_{\text{slope}}x + \underbrace{b}_{\text{y-intercept}}$

→ Quadratic function: $f(x) = ax^2 + bx + c$

→ Cubic function: $f(x) = ax^3 + bx^2 + cx + d$

→ Polynomial function: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$

w/ – n : nonnegative integer
– a_n : coefficients (degree d: $\forall n \in \mathbb{N}^*, d \geq n$)

3. Types of functions

- Power functions x^n $(n \in \mathbb{N}^*)$
- Root functions $x^{\frac{1}{n}}$ $(n \in \mathbb{N}^*)$

⊣ Reciprocal function x^{-1} $(x \neq 0)$ ⊢

- Rational functions
- Algebraic functions

- Trigonometric functions

prin $\begin{cases} \sin \\ \cos \end{cases}$ cosec (csc), sec (sec)

Ratio $\{ \tan = \frac{\sin}{\cos}$ $\underbrace{\text{cotan} = \frac{\cos}{\sin}}_{\text{Reciprocal : } \frac{1}{x}}$ (cot)

inverse : f^{-1} arcsin, arccos, arctan, arccsc, arcsec, arccot

- Exponential functions $f(x) = a^x$
- Logarithmic functions $f(x) = \log_a x$ } base : a
- Transcendental functions $\neq$ algebraic

4. Opera°

- $(f+g)(x) = f(x) + g(x)$, etc..
- $f \circ g(x) = f(g(x))$ "f circle g"
- Laws of exponents: $a^x a^y = a^{x+y}$ $(a^x)^y = a^{xy}$
  $a^x / a^y = a^{x-y}$ $(ab)^x = a^x b^x$

5. Inverse functions

- One-to-one function: $\forall (x_1, x_2) \in D_f^2 (x_1 \neq x_2), f(x_1) \neq f(x_2)$

→ Horizontal line test: A func° is one-to-one if and only if no horizontal line intersects its graph more than once

- Inverse func° of f w/ domain A and range B is f^{-1} with domain B and range A
- ⚠ $\underbrace{f^{-1}(x)}_{\text{→ inverse (fr: récipr°)}} \neq \underbrace{[f(x)]^{-1}}_{\text{→ reciprocal (fr: inverse)}}$
- Laws of logarithms: $\log_a(xy) = \log_a x + \log_a y$
  $\log_a(x/y) = \log_a x - \log_a y$
  $\log_a(x^R) = R \log_a(x)$
- Change of base: $\log_a x = \frac{\ln x}{\ln a}$
- $\sin^{-1}(x) = y \Leftrightarrow \sin y = x$ and $y \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

↳ The domain is restricted (for trig func°)

Chapter 02: Limits and derivatives

→ Asymptotes (vertical)

if $\lim_{x \to a} f(x) = \pm\infty$, then $x = a$ is a vertical asymptote of the curve $y = f(x)$

→ Limit Laws

- $\lim_a f + g = \lim f + \lim g$, etc. ($+, c*, *, \div, e^x$)
- $\lim_{x \to a} c = c$, $\lim_{x \to a} x = a$, $\lim_{x \to a} x^n = a^n$
- Limits in a of polynomial or rational f°: $\lim_{x \to a} f(x) = f(a)$
- $\lim_{a^-} f(x) = L$ and $\lim_{a^+} f(x) = L \iff \lim_a f(x) = L$

→ Limit theorems

- if $f(x) \leq g(x)$ when x is near a: $\lim_a f \leq \lim_a g$
- Squeeze theorem: $f(x) \leq g(x) \leq h(x)$ when x is near a and $\lim_a f = \lim_a h$, then $\lim_a g = \lim_a f = \lim_a h$

→ Defini° of a limit

Let f be a func° defined on some open interval that contains the nbr a, except possibly a itself. Then we say that the limit of $f(x)$ as x approaches a is L, and we write:

$\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

- Left-hand limit:

$$\lim_{x \to a^-} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0, (a - \delta < x < a) \Rightarrow |f(x) - L| < \varepsilon$$

- Right-hand limit:

$$\lim_{x \to a^+} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0, (a < x < a + \delta) \Rightarrow |f(x) - L| < \varepsilon$$

- Infinite limit:

$$\lim_{x \to a} f(x) = +\infty \iff \forall M > 0, \exists \delta > 0, (0 < |x - a| < \delta) \Rightarrow f(x) > M$$

→ Continuity

- f is continuous <u>at</u> a if $\lim_{x\to a} f(x) = f(a)$
- All polynomial func° are continuous on $\mathbb{R}$
  All Rational func° are continuous on the domain on which they are defined
- $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$

→ The intermediate value theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a nbr c in (a, b) such that $f(c) = N$

→ Asymptotes (horizontal)

if $\lim_{x\to\pm\infty} f(x) = a$, then $y = a$ is a horizontal asymptote of the curve $y = f(x)$

→ Limits of usual functions

- if R is a strictly positive rational nbr, then $\lim_{x\to\pm\infty} \frac{1}{x^R} = 0$

→ Derivatives

- The tangent line to the curve $y = f(x)$ at $P\,(a, f(a))$ is the line through P with slope $m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$

  $m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$
- The derivative of f at a is $f'(a) = m$

  ⇒ the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$

→ Derivative functions

- $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

- Note: if f is differentiable at a, f is continuous at a

→ Higher derivatives

- $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

- Note: differentiation operators: $\left.\frac{dy}{dx}\right|_{x=a} = f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

Chapter 03: Differentiation rules

→ Polynomials

- $\frac{d}{dx}(c)=0$, $\frac{d}{dx}(x)=1$, $\frac{d}{dx}(x^n)=nx^{n-1}$ $\binom{n\in\mathbb{N}^{*+}}{n\in\mathbb{R}}$
- Const^N multiple rule : $\frac{d}{dx}(c\,f(x)) = c\frac{d}{dx}f(x)$, etc...

→ Exponentials

- e is the nb such that $\lim_{h\to 0}\frac{e^h-1}{h}=1$
- $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}(a^x) = a^x \ln a$
- Product rule : $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$
- Quotient rule : $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$

→ Trigonometric

- $\lim_{x\to 0}\frac{\sin x}{x}=1$
- $\lim_{x\to 0}\frac{\cos x - 1}{x}=0$

→ $\frac{d}{dx}\sin x = \cos x$
→ $\frac{d}{dx}\cos x = -\sin x$

→ Chain rule :

- $(f\circ g)' = f'(g)\times g' \Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
- ↳ $\frac{d}{dx}[g(x)]^n = n(g(x))^{n-1}\times g'(x)$

→ Implicit differentiation

- $x^a + y^b = c \to \frac{d}{dx}(x^a+y^b) = \frac{d}{dx}c \to \frac{d}{dx}x^a + \frac{d}{dy}(y^b)\frac{dy}{dx}=0$

$\to ax^{a-1} + by^{b-1}\times\frac{dy}{dx}=0 \to \frac{dy}{dx} = -\frac{ax^{a-1}}{by^{b-1}}$

→ Logarithms

- $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$
- $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)} = \frac{1}{f(x)}f'(x)\left(=\frac{1}{f}\frac{df}{dx}\right)$

→ Logarithmic differentiation

Method	example
1. take ln on both sides	$y = x^n$
	$\ln(y) = n \ln(x)$
2. implicitely differentiate	$\frac{y'}{y} = \frac{n}{x}$
3. solve equa° for y'	$y' = \frac{n}{x} y = n \times x^{-1} x^n$
	$y' = n x^{n-1}$

Chapter 04: Applications of differentiation

→ Fermat's theorem

- f has a local extremum (min/max) at c, and f' (exists), then $f'(c) = 0$

→ Extreme value theorem

- f cont. on a closed interval $\Rightarrow f$ attains absolute extremum values

→ Extremums

- Critical nb of a func° is a nb c in the domain of f such that either $f'(c) = 0$, or $f'(c)$ does not exist.
- Closed interval method $[a, b]$
  - → Find values of f at critical nbs of f in (a, b)
  - → Find values of f at end points
  - → The smallest and largest values found are resp. the absolute min and max.

→ The mean value theorem

- Rolle's theorem

$$\left.\begin{array}{l} - f \text{ is continuous on } [a,b] \\ - f \text{ is differentiable on } (a,b) \\ - f(a) = f(b) \end{array}\right\} \Rightarrow \exists c \in (a,b),\ f'(c) = 0$$

- The mean value theorem
  - f is cont. on $[a, b]$
  - f is diff. on (a, b)

$$\Rightarrow \exists c \in (a,b), f'(c) = \frac{f(b) - f(a)}{b - a}$$

→ The first derivative test

- if f' changes
  - from ⊕ to ⊖ in c, c is a local max
  - from ⊖ to ⊕ in c, c is a local min

→ Concavity Test

- if $f''(x) > 0$, $\forall x \in I$, the graph of f is concave upward on I
- if $f''(x) < 0$, $\forall x \in I$, " downward on I

→ An <u>inflection point</u> P of a curve $y = f(x)$ is a point where the curve changes the direction of its concavity ~

→ The second derivative test

- if $f'(c) = 0 \wedge f''(x) > 0$, then f has a local min at c
- if $f'(c) = 0 \wedge f''(x) < 0$, " max at c

→ L'Hospital's rule

- If f and g are derivable and $g'(x) \neq 0$ on an open interval I and either $\lim_a f = \lim_a g = 0$ or $\lim_a f = \lim_a g = \pm\infty$,

Indeterminate forms

then $\lim_a \frac{f(x)}{g(x)} = \lim_a \frac{f'(x)}{g'(x)}$ if the lim exists or is $\pm\infty$.

→ Absolute extreme values

- if c is a critical value and $\forall x < c, f'(x) > 0$ and $\forall x > c, f'(x) < 0$, then c is the absolute max value of f
- if c " $\forall x < 0, f'(x) <$ " $\forall x > c, f'(x) > 0$, then c is the absolute min value of f

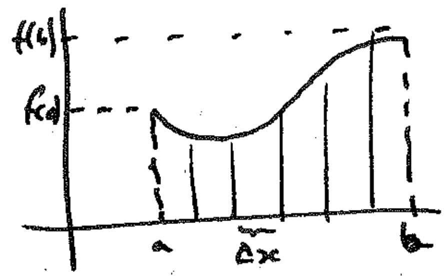
→ Antiderivatives

Def: $\forall x \in I$, $F'(x) = f(x) \Leftrightarrow F$ is an antiderivative of f on I

Chapter 05 : Integrals

→ Area under a curve

- $\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i)\Delta x$



→ Definite integral

- $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

→ Midpoint rule

- $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x \qquad \left(\begin{array}{l} \Delta x = \frac{b-a}{n} \\ \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) \end{array} \right)$

→ Properties of the integral

- $\int_a^b c\,dx = c(b-a)$
- $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_c^b g(x)\,dx$
- $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

→ Comparison properties of the integral

- if $\forall x \in [a,b]$, $f(x) \geq 0$, then $\int_a^b f(x)\,dx \geq 0$
- if $\forall x \in [a,b]$, $f(x) \geq g(x)$, then $\int_a^b f(x)\,dx \geq \int_c^b g(x)\,dx$
- if $\forall x \in [a,b]$, $f(x) \in [m,M]$, then $\int_a^b f(x)\,dx \in [m(b-a), M(b-a)]$

→ The Fundamental Theorem of Calculus

- f continuous on $[a,b] \Rightarrow \int_a^x f(t)\,dt$ $(x \in [a,b])$ is continuous and differentiable on (a,b) and $\left(\int_a^x f(t)\,dt\right)' = f(x)$

- f continuous on $[a,b] \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$ (where $F'(x) = f(x)$)

$$f \text{ cont } [a,b] : \begin{cases} g(x) = \int_a^x f(t)\,dt \Rightarrow g(x)' = f(x) \\ \int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F'(x) = f(x) \end{cases}$$

→ The substitution rule

- if $u = g(x)$ (w/ range I) and u is dif. and f is cont. on I, then $\forall x \in I = [a,b]$, $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

→ Symmetry and integrals

- if f is even and cont. on $[-a,a]$: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
- if f is odd " : $\int_{-a}^{a} f(x)\,dx = 0$

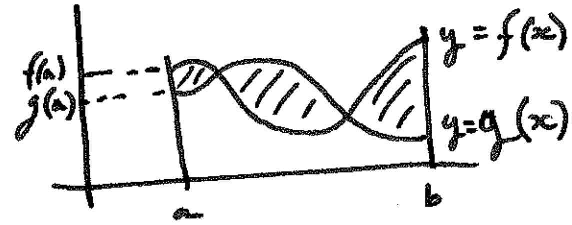
→ Link between dif/int rules

dif			int
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $(f \circ g)' = f'(g) \cdot g'$	Chain Rule	Substitution	with $F' = f$, $F(u) = \int f(u)\,du$ and $F(u) = \int \frac{dF(u)}{dx}\,dx = \int \frac{d}{du}F(u)\frac{du}{dx}\,dx$ $\int f(u)\,du = \int f(u)\,u'\,dx$
$\frac{d}{dx} f \cdot g = f\frac{d}{dx}g + g\frac{d}{dx}f$ $(f \cdot g)' = f'g + g'f$	Product Rule	Integration by parts	$fg = \int \frac{d}{dx} f \cdot g\,dx = \int (f'g + g'f)\,dx$ $f \cdot g = \int f'g\,dx + \int g'f\,dx$ $\Rightarrow \int f'g\,dx = f \cdot g - \int g'f\,dx$ $\Rightarrow \int g\,df = fg - \int f\,dg$

Chapter 06: Applica° of integration

→ Area between curves

- $A = \int_a^b |f(x) - g(x)|\,dx$



→ The mean value theorem for integrals

- f cont. $[a,b] \Rightarrow \exists c \in [a,b],\ f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\Rightarrow \int_a^b f(x)\,dx = f(c)\,(b-a)$$

Chapter 07: Techniques of integration

→ Integration by parts

- f' and g' cont. $[a,b]$: $\int_a^b f(x)g'(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$

→ Evaluating trigonometric integrals

To evaluate $\int \sin^m x \cos^n x\,dx$

- if $n = 2k+1$ (power of cosine is odd), use $\cos^2 x = 1 - \sin^2 x$:

$$\int \sin^m x \cos^{2k+1} x\,dx = \int \sin^m x\,(\cos^2 x)^k \cos x\,dx$$
$$= \int \sin^m x\,(1-\sin^2 x)^k \cos x\,dx$$

- if $m = 2k+1$ (power of sine is odd), use $\sin^2 x = 1 - \cos^2 x$:

$$\int \sin^{2k+1} x \cos^n x\,dx = \int (1-\cos^2 x)^k \cos^n x \sin x\,dx$$

then substitute $u = \cos x$

- if m and n are even, use : $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$
$\sin x \cos x = \frac{1}{2}\sin 2x)$

→ Improper integrals of type 1

. if $\forall t \geq a$, $\int_a^t f(x)\,dx$ exists then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

. if $\forall t \leq a$, $\int_t^a f(x)\,dx$ exists then $\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$

If both of these limits exist: (as finite numbers) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$

→ Improper integrals of type 2

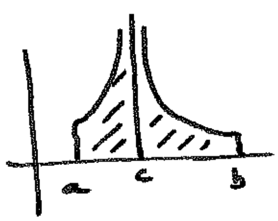
. if f cont $[a,b)$ and discontinuous at b, then:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

. if f cont $(a,b]$ and discontinuous at a, then:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

If both limits exist (as finite numbers) for f cont on $[a,c)$ and cont on $(c,b]$



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

→ Comparison theorem

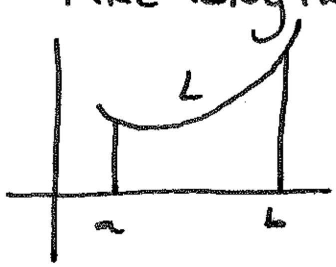
For f and g cont on $[a,\infty)$ and $\forall x \geq a$, $f(x) \geq g(x) \geq 0$

$\int_a^\infty f(x)\,dx$ is convergent $\Rightarrow \int_a^\infty g(x)\,dx$ is converg.

$\int_a^\infty g(x)\,dx$ is divergent $\Rightarrow \int_a^\infty f(x)\,dx$ is diverg.

Chapter 08: further applica° of integration

→ Arc length



f' cont $[a,b] \Rightarrow L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$

LEC01 – Parametric curves

→ Tangents

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \left(\frac{dx}{dt} \neq 0\right) \qquad \begin{cases} y = \frac{dy}{dx} x + b \\ b = y_t - \frac{dy}{dx_t} \end{cases}$$

→ Areas

$$A = \int_a^b y(x)\,dx = \int_\alpha^\beta y(x)\,x'(t)\,dt = \int_\alpha^\beta y(t)\,x'(t)\,dt \quad \left(\text{where } \begin{cases} x(\beta) = b \\ x(\alpha) = a \end{cases}\right)$$

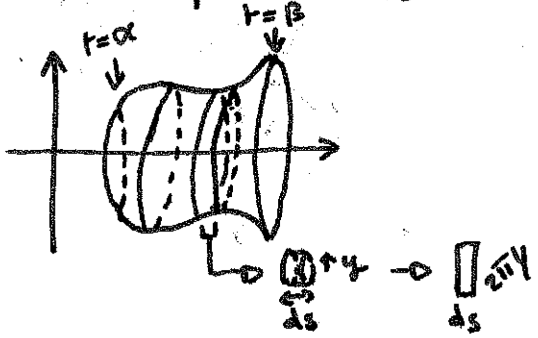
→ Arc Length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \Rightarrow \begin{cases} L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ \frac{dx}{dt} \neq 0 \end{cases}$$

(same ↗ for α, β)
(note: $t \in [\alpha, \beta]$)

Note: ensure C is traversed exactly once as t increases from α to β.

→ Surface Area

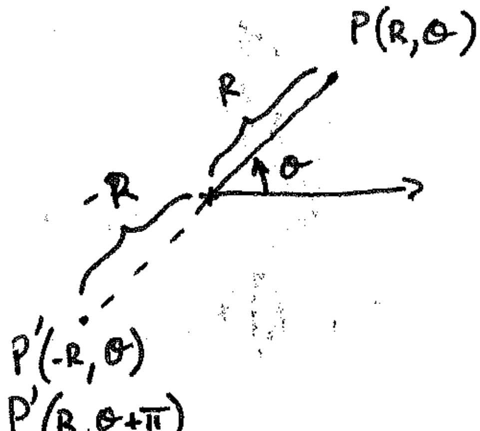


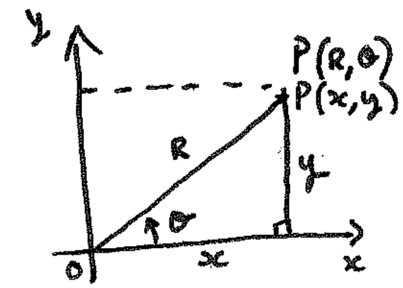
$$S = \sum^{\infty} 2\pi y\,ds = \sum^{\infty} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\Rightarrow S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

LEC02 – Polar curves

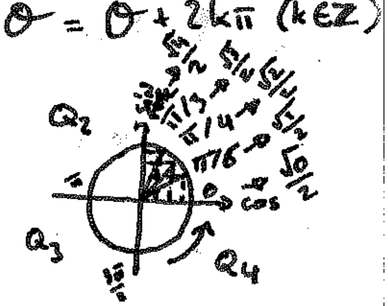
→ Polar coordinate system





$$\hookrightarrow \begin{rcases} \cos\theta = \frac{x}{R} \\ \sin\theta = \frac{y}{R} \end{rcases} \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases}$$

$\theta \equiv \theta \; [2\pi]$
$\Leftrightarrow \theta = \theta + 2k\pi \quad (k \in \mathbb{Z})$

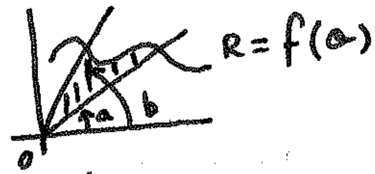


→ Tangents to polar curves

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \qquad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{R'\sin\theta + R\cos\theta}{R'\cos\theta - R\sin\theta}$$

→ Areas in polar coordinates

$$A = \int_a^b \frac{1}{2}\underbrace{[f(\theta)]^2}_{R^2}\, d\theta$$

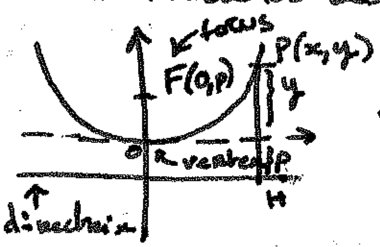


→ Arc length in polar coordinates

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{R^2 + \left(\frac{dR}{d\theta}\right)^2}\, d\theta$$

LEC 03 - Conic sections

→ Parabolas



$|PF| = |PH|$

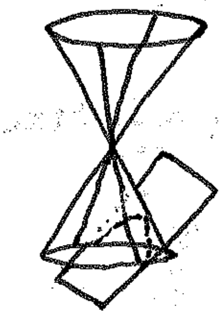
$$\sqrt{x^2 + (y-p)^2} = y + p$$

$$x^2 = 4py$$

$$\left(\frac{|PF|}{|PL|} = e \right)$$

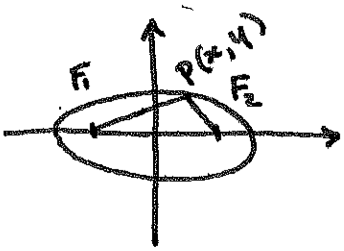
focus, directrix

e : eccentricity



e = 1

→ Ellipses



$$|PF_1| + |PF_2| = 2a$$

$$\sqrt{(x-c)^2 + y^2} = -\sqrt{(x+c)^2 + y^2} + 2a$$

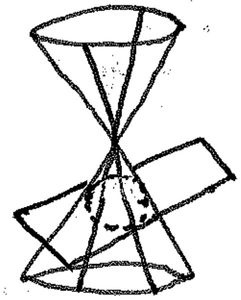
square

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

square

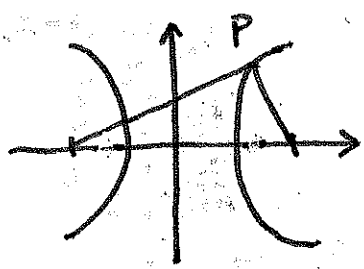
$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

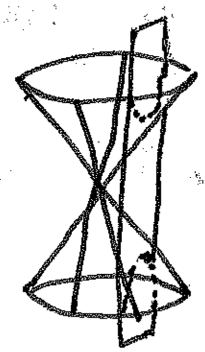


e < 1

→ Hyperbolas



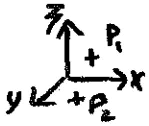
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



e > 1

LEC04 - Vectors

→ xyz-plane

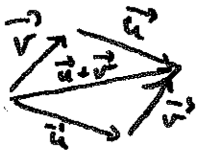


$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$\begin{cases} P(x,y,z) \\ C(a,b,c) \\ \|PC\| = R \end{cases} \Rightarrow (x-a)^2+(y-b)^2+(z-c)^2 = R^2$

$(\text{if } (a,b,c) = (0,0,0),\ x^2+y^2+z^2 = R^2$

→ vectors

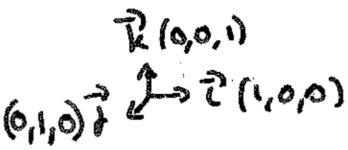


$\vec{AB} = \langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$ → components

$|\vec{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$ → magnitude

Operations → $c\vec{a} + d\vec{b} = \langle ca_1+db_1, ca_2+db_2, ca_3+db_3\rangle$

(+) →
- (a) associativity - $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$
- (c) commutativity - $\vec{a}+\vec{b} = \vec{b}+\vec{a}$
- (l) linearity - $c\vec{a}+d\vec{b}$
- (d) distributivity - $c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$ // $\vec{a}(c+b) = c\vec{a}+b\vec{a}$

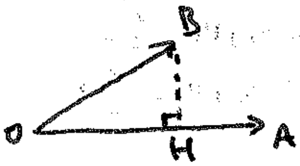


Unit vector
→ $\|\vec{u}\| = 1$

Normalized vector
→ $\dfrac{\vec{a}}{\|\vec{a}\|}$

| (N) Neutral element - $\vec{0} = \vec{a}-\vec{a}$
| ((T) transitivity - $a \angle b \wedge b \angle c \Rightarrow a \angle c$))

→ Dot product



$\vec{OA}\cdot\vec{OB} = \overline{OA}\times\overline{OH}$ $\boxed{1}$

$\vec{u}\cdot\vec{v} = x_1x_2+y_1y_2+z_1z_2$ $\boxed{2}$

$\vec{u}\cdot\vec{v} = |\vec{u}|\times|\vec{v}|\times\cos(\vec{u},\vec{v})$ $\boxed{3}$

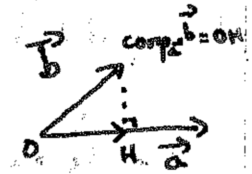
$\vec{u}\cdot\vec{v} = \frac{1}{2}\left(|\vec{u}+\vec{v}|^2-|\vec{u}|^2-|\vec{v}|^2\right)$ $\boxed{4}$

Rules
$\vec{u}\cdot\vec{u} = |\vec{u}|^2$
$\vec{u}\cdot\vec{0} = 0$
(a), (c), (d), (l), (N)

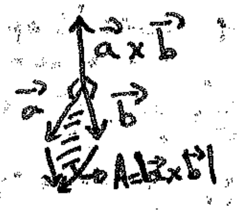
Direction angles: $(\vec{a},\vec{i}) \in [0,\pi]$, $\cos(\vec{a},\vec{i}) = \dfrac{a_1}{|a|}$ $\left(\dfrac{\vec{a}\cdot\vec{i}}{|\vec{a}|\times|\vec{i}|}\right)$

Projections
→ scalar: $\text{comp}_{\vec{a}}\vec{b} = \underbrace{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$

→ vector: $\text{proj}_{\vec{a}}\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$



→ Cross product



$\vec{a}\times\vec{b} = |\vec{a}|\times|\vec{b}|\times\sin(\vec{a},\vec{b})\times\vec{n}$ $(|\vec{n}| = 1 \wedge \vec{n}\cdot\vec{a} = \vec{n}\cdot\vec{b} = 0)$

$\vec{a}\times\vec{b} = \det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{pmatrix} = \det\begin{pmatrix}a_2 & a_3\\ b_2 & b_3\end{pmatrix}\vec{i} - \det\begin{pmatrix}a_1 & a_3\\ b_1 & b_3\end{pmatrix}\vec{j} + \det\begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}\vec{k}$

$= (a_2b_3-b_2a_3)\vec{i} - (a_1b_3-b_1a_3)\vec{j} + (a_1b_2-b_1a_2)\vec{k}$

$\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$

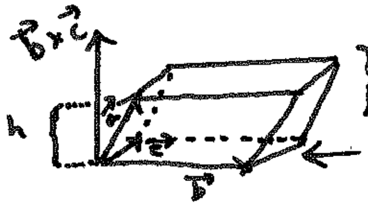
$(\vec{a}\cdot\vec{b}) = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

rules: (a), $\cancel{(c)}$, (d), (l), (N)

→ Triple product

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} \qquad (\text{Note}: (\vec{a}\times\vec{b})\cdot\vec{a} = 0)$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

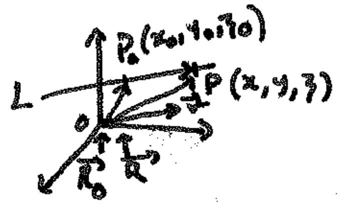


$$V = A\times h = |\vec{b}\times\vec{c}|\times(|\vec{a}|\times\cos(\vec{a},\vec{b}\times\vec{c})) = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

$$A = |\vec{b}\times\vec{c}|$$

LEC 05 - Equations in space

→ Lines

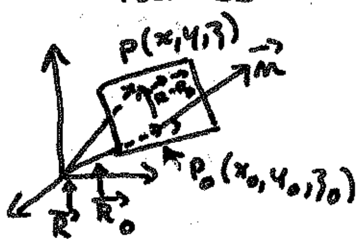


$$\vec{R} = \vec{R_0} + t\vec{v} \Rightarrow \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \Rightarrow t = \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$\vec{R_0}$: vec. to fixed point; $\vec{v}$: vec. $\parallel L$ ($\vec{v} = \langle a,b,c\rangle$)

vector — parametric — symmetric

→ Planes



$$\vec{n}\cdot(\vec{R}-\vec{R_0}) = 0$$

$$\vec{n}\cdot\vec{R} = \vec{n}\cdot\vec{R_0} \Rightarrow a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

$$\vec{n} = \langle a,b,c\rangle \xleftrightarrow{\text{link!}} (ax+by+cz = d,\ d = ax_0+by_0+cz_0)$$

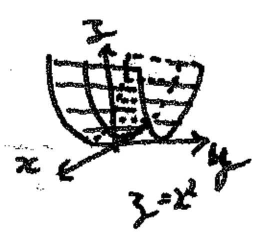
Angle between planes: $P_1: \alpha_1 x + \beta_1 y + \gamma_1 z = \delta_1$
$P_2: \alpha_2 x + \beta_2 y + \gamma_2 z = \delta_2$ $\Rightarrow \vec{n_1} = \langle\alpha_1,\beta_1,\gamma_1\rangle$, $\vec{n_2} = \langle\alpha_2,\beta_2,\gamma_2\rangle$

$$\text{ang}(P_1,P_2) = (\vec{n_1},\vec{n_2})$$

$$\to \cos(\vec{n_1},\vec{n_2}) = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\times|\vec{n_2}|}$$

→ Cylinders

Cylinder: surface consisting of ∥ lines (rulings) which pass through a given <u>plane</u> curve



→ Quadric surfaces

Counterparts in 3D of conic sections.
Graph of 2nd degree equa° with 3 variables.

ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	elliptic paraboloid $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	hyperboloid of one sheet $+\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
cone $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	hyperbolic paraboloid $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	hyperboloid of two sheets $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

LEC 06 - Vector functions

→ Vector calculus

$\forall t \in D_f \cap D_g \cap D_h \,,\ \vec{R}(t) = \langle f(t), g(t), h(t) \rangle$

- $\lim_{t \to t_0} \vec{R}(t) = \langle \lim_{t_0} f, \lim_{t_0} g, \lim_{t_0} h \rangle$
- $R'(t) = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t} = \langle f'(t), g'(t), h'(t) \rangle$ Rules: ⊕, chain rule, product rule
  - ↳ $*$ ✓
  - ↳ $\cdot$ ✓
  - (↳ $\times$ ✓⚠ non-commut)
- $\int_a^b \vec{R}(t) = \left[\int_a^b f(t)\,dt\right]\vec{i} + \left[\int_a^b g(t)\,dt\right]\vec{j} + \left[\int_a^b h(t)\,dt\right]\vec{k}$

→ Arc length & curvature

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b |R'(t)|\,dt$$

Arc length function: $s(t) = \int_a^t \sqrt{x'^2(u) + y'^2(u) + z'^2(u)}\,du = \int_a^t |R'(u)|\,du$

$t \in [a,b]$

as t increases, C is traversed exactly once

$\boxed{s'(t) = |R'(t)|}$

Curvature: $K = \left|\frac{d\vec{T}}{ds}\right|$ where $\vec{T}$ is unit tangent vector to $\vec{R}$

$$T(t) = \frac{R'(t)}{|R'(t)|}$$

$$K(t) = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \left|\frac{T'(t)}{R'(t)}\right|$$

→ Reminder : synthetic division

Find obvious root : $2x^2 - 5x - 12$

Divide by (x-root): $(x+4) \,|\, \frac{2x^2 - 5x - 12}{2x - 3}$

biproduct → $8x$

what to product → $-3x$

Division algorithm : $196.00 \,|\, 16$ → 12.25

36

4.0

$.80$

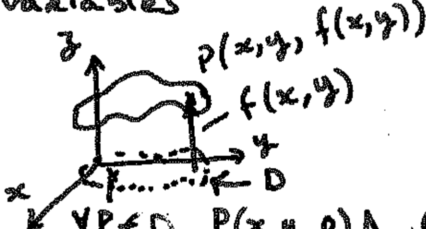
0

LEC 07 - Functions of several variables

→ Functions of 2 variables

$D_{f(x,y)} \subset \mathbb{R}^2$

$D_f = (D_x \cap D_y)^2$



$\forall P \in D,\ P(x,y,0) \wedge (x,y) \in D_f$

Level curves / sets:
$f(x,y) = k$

→ Limits & continuity

$\lim\limits_{(x_1 \dots x_n) \to (a_1 \dots a_n)} f(x_1 \dots x_n) = L \iff \forall \varepsilon > 0, \exists \delta > 0, (x,y) \in D_f \wedge 0 < \sqrt{(x_1 - a_1)^2 + \dots} < \delta$
$\implies |f(x,y) - L| < \varepsilon$

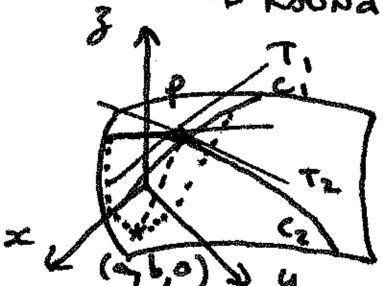
f is continuous at $(a_1 \dots a_n) \iff \begin{cases} (a_1 \dots a_n) \in D_f \\ f(a_1 \dots a_n) = \lim\limits_{(x_1 \dots x_n) \to (a_1 \dots a_n)} f(x_1 \dots x_n) = L \end{cases}$

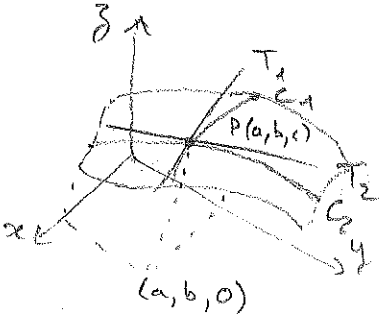
Note: to show f is not cont. → 1. show $\nexists \lim$
2. if $\exists \lim$, show $f(\dots) \neq \lim\limits_{(\dots)} f$

LEC 08 - Partial derivatives

→ Partial derivatives

$$f_x = \underbrace{\frac{\partial f}{\partial x}}_{\text{D'ROUND d'}} = \frac{\partial}{\partial x} f(x,y) = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \left(= \frac{\partial z}{\partial x} \right)$$





$C_1: f(x, b)$
→ $T_1: f_x(a,b)$
$C_2: f(a, y)$
→ $T_2: f_y(a,b)$

→ Higher derivatives

"Second partial derivative with respect to x of 'f sub x' (f_x)"

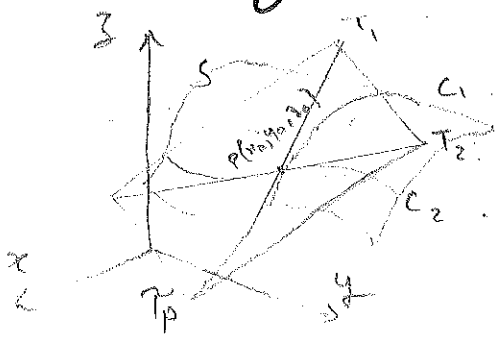
$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

"Second partial derivative with respect to y of 'f sub x' (f_x)"

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

LEC09 - Differentiability of a multivariable function

→ Tangent planes



τ_p passes through $P(x_0, y_0, z_0)$, thus:

- $z - z_0 = a(x - x_0) + b(y - y_0)$
- $T_1 \subset \tau_p \Rightarrow \begin{cases} y = y_0 \\ z - z_0 = a(x - x_0) \end{cases}$

$\Rightarrow a = f_x(x_0, y_0) \rightarrow$ slope of T_1

$T_2 \subset \tau_p \Rightarrow b = f_y(x_0, y_0) \rightarrow$ slope of T_2

Equation of tang^n plane at $P(x_0, y_0, z_0)$ of surface $z = f(x,y)$:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

→ Linear approximations

- Linearization of $f(x,y)$ at (a,b) is the tang^n plane at (a,b):

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

- Linear approxima° to f at (a,b): $f(x,y) \approx L(x,y)$
- (CALC I: we know f diff at $a \Leftrightarrow \Delta y = f'(a)\Delta x + \varepsilon \Delta x$ w/ $\lim\limits_{\Delta x \to 0} \varepsilon = 0$)

CALC II: $f(x,y)$ diff at $(a,b) \Leftrightarrow \Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ w/ $\lim\limits_{\Delta x, \Delta y \to 0,0} \varepsilon_1 = \lim\limits_{\Delta x, \Delta y \to 0,0} \varepsilon_2 = 0$

- f_x and f_y exist <u>near</u> (a,b) and are continuous $\Rightarrow f$ is diff at (a,b)

→ Total / Full differential

- (CALC I: $dy = f'(x)\,dx$)

CALC II:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

same

$$df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy$$

→ Functions of 3 or more variables

- Same rules ($dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$ etc.)
- Level curves for graphs 3D $\Leftrightarrow$ level surfaces for graphs 4D!

(• Ink shifted from black → blue)

→ The Chain Rule

- Chain rule I

$z = f(x,y)$, $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

f is diff, x is diff, y is diff $\Rightarrow \dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

- Chain rule II

$z = f(x,y)$, $\begin{cases} x = x(s,t) \\ y = y(s,t) \end{cases}$

$f/x/y$ are diff $\Rightarrow \begin{cases} \dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} \\ \dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} \end{cases}$ $\left| \begin{array}{l} \dfrac{\partial f}{\partial s} = f_x x_s + f_y y_s \\ \dfrac{\partial f}{\partial t} = f_x x_t + f_y y_t \end{array} \right.$

- Chain rule (general version)

$z = f(x_1 \ldots x_n)$, $\begin{cases} x_1 = x_1(t_1 \ldots t_m) \\ \ldots \end{cases}$

everything is diff $\Rightarrow \dfrac{\partial z}{\partial t_i} = \dfrac{\partial z}{\partial x_1}\dfrac{\partial x_1}{\partial t_i} + \ldots + \dfrac{\partial z}{\partial x_n}\dfrac{\partial x_n}{\partial t_i}$

→ Clairaut's theorem

$(a,b) \in D_f \land f_{xy}$ and f_{yx} are cont. on $D_f \Rightarrow f_{xy}(a,b) = f_{yx}(a,b)$

Proof: let $g(x) = f(x, b+h) - f(x,b)$

- $\Delta h = g(a+h) - g(a)$

Mean Value Thm: $\exists c \in [a, a+h], hg'(c) = g(a+h) - g(a)$

$\Rightarrow \Delta h = h\left[f_x(c, b+h) - f_x(c,b)\right]$

Mean Value Thm: $\exists d \in [b, b+h], h f_{xy}(c,d) = f_x(c,b+h) - f_x(c,b)$

$\Rightarrow \Delta h = h^2 f_{xy}(c,d)$

Continuity of f_{xy} at (a,b):

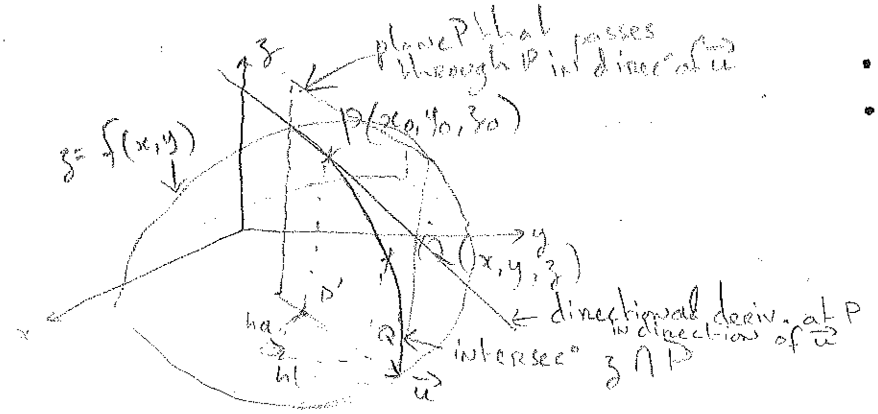
$\lim_{h \to 0} \dfrac{\Delta h}{h^2} = \lim_{c,d \to a,b} f_{xy}(c,d) = f_{xy}(a,b)$

- Similarly: $\lim_{h \to 0} \dfrac{\Delta h}{h^2} = f_{yx}(a,b)$

- Thus: $f_{xy}(a,b) = f_{yx}(a,b)$

LEC 10 - Gradient and directional derivative

→ Directional derivative



- $\vec{u}$ is a unit vector in the xy-plane
- $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$ are the proj of P and Q in the xy-plane

$$\Rightarrow \overrightarrow{P'Q'} = h\vec{u} = \langle ha, hb \rangle$$

$$\Rightarrow \begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases}$$

The direc° deriv. of f at (x_0, y_0) in the direction of unit vector $\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Since $D_{\vec{i}} f(x_0, y_0) = f_x(x_0, y_0)$ and $D_{\vec{j}} f(x_0, y_0) = f_y(x_0, y_0)$

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

→ Gradient vector

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \langle f_x(x,y), f_y(x,y) \rangle$$

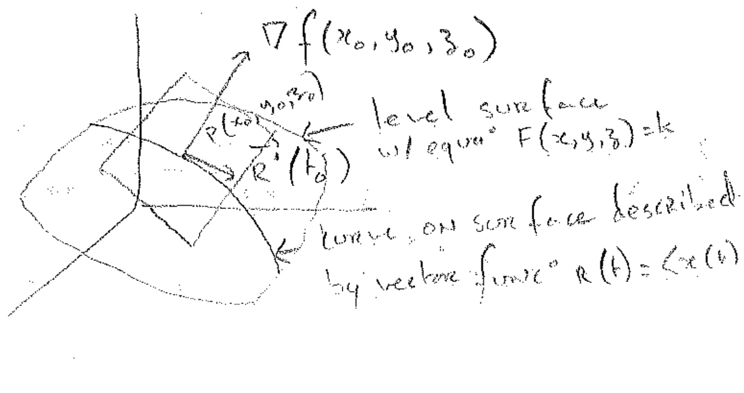
'nabla' to read: " del f "

$$\Rightarrow D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

- Max value of $D_{\vec{u}} f$: $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| . |\vec{u}| . \cos\theta = |\nabla f| \cos\theta$

Thus $D_{\vec{u}} f$ is maximized when $\cos\theta = 1$ $(|\vec{u}| = 1)$ (i.e. $\theta = 0$)

→ Tangent planes to level surfaces



$$F(x(t), y(t), z(t)) = k$$

chain rule

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

Since $\begin{cases} \nabla f = \langle F_x, F_y, F_z \rangle \\ \vec{R}'(t) = \langle x'(t), y'(t), z'(t) \rangle \end{cases}$

$$\rightarrow \nabla f \cdot R'(t) = 0 \quad (\nabla f \perp \vec{R}')$$

LEC11 - Critical points, min and max

→ Local max/min (critical pts)

$$\begin{cases} (a,b) \text{ is a local max/min} \\ f_x \wedge f_y \text{ exist} \end{cases} \Leftrightarrow \begin{cases} f_x(a,b) = 0 \\ f_y(a,b) = 0 \end{cases}$$

→ Second derivatives test

(a,b) is a critical pt $\left(\begin{matrix} f_x(a,b)=0 \\ f_y(a,b)=0 \end{matrix}\right)$

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

→ . $D > 0 \wedge f_{xx}(a,b) > 0 \Rightarrow f(a,b)$ is a local min

. $D > 0 \wedge f_{xx}(a,b) < 0 \Rightarrow f(a,b)$ is a local max

. $D < 0 \Rightarrow \begin{cases} f(a,b) \text{ is not a critical point.} \\ (a,b) \text{ is a saddle point of } f \end{cases}$ (graph of f crosses its tangent plane at (a,b))

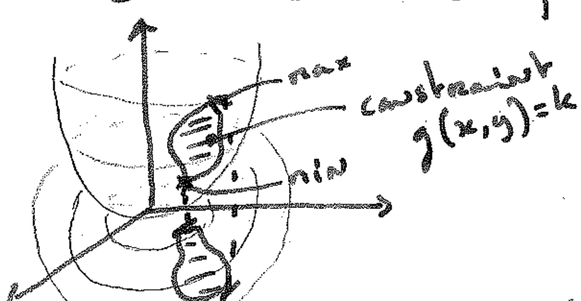
→ Absolute max/min

. Extreme value theorem: f cont on closed, bounded set $D \subset \mathbb{R}^2$
$\Rightarrow \exists$ abs max $\wedge$ $\exists$ abs min

. Abs max = max (local maximums, f(boundaries of D))
Abs min = min (local minimums, f(boundaries of D))

LEC12 - Method of Lagrange multipliers.

→ Extreme values of func° of 2 vars under 1 constraint.



These max and min pts are the unique pts where the tangent lines of g and f are the same.

$$\Rightarrow \nabla f = \lambda \nabla g$$

where λ is a scalar called 'Lagrange Multiplier'

→ Method of λ

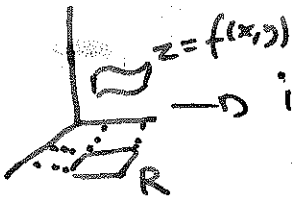
To find the min/max values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ (assuming $\nabla g \neq 0$): → method cannot be applied in such cases

① Find critical pts of $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ ← Lagrange func°
by solving $\begin{cases} L_x = 0 \\ L_y = 0 \\ L_\lambda = 0 \end{cases}$

② compare to get max/min

LEC13 - Average value of a func°, double $\int$, Fubini's theorem

- Double Riemann Sum : $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta A$$

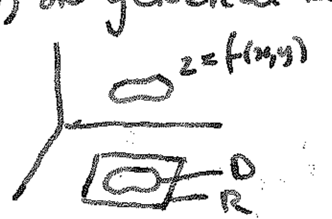


$\longrightarrow$ if $f(x,y) \geq 0$, $\iint_R f(x,y)\,dA$ describes the volume of the solid between R and $z = f(x,y)$ –

- Average value :
  - $\bar{f} = \dfrac{1}{b-a} \int_a^b f(x)\,dx$
  - $\bar{f} = \dfrac{1}{\iint_D 1\,dA} \iint_D f(x,y)\,dA$
  - $\bar{f} = \dfrac{1}{\iiint_E 1\,dV} \iiint_E f(x,y,z)\,dV$

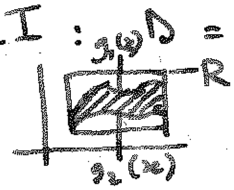
- Iterated integrals : partial integra° (inverse process of partial dif.)
- Fubini's theorem : $$\iint_{R=[a,b]\times[c,d]} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

- $\iint$ on general regions :



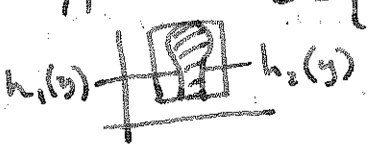
$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA \quad \left(F(x,y) = \begin{cases} f(x,y), & (x,y)\in D \\ 0, & (x,y)\notin D \end{cases} \right.$$

.Type I : $D = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$



$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

. Type II : $D = \{(x,y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$



$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- Changing the order of integra° : redefining the domain as type I/II.

e.g. : $D = \{(x,y) \mid 0 \leq x \leq 2,\ x^2 \leq y \leq 2x\}$
$= \{(x,y) \mid 0 \leq y \leq 4,\ \frac{1}{2}y \leq x \leq \sqrt{y}\}$

LEC 14 - Double integrals in polar coordinates

- $$\iint_{D_{xy}} f(x,y)\, dA_{xy} = \iint_{D_{R\theta}} f(R\cos\theta, R\sin\theta)\, R\, dA_{R\theta}$$

$\left(dA_{xy} = R\, dA_{R\theta} \right)$

polar rectangle ($\frac{1}{2}R^2\theta$)

polar rectangle $\longrightarrow$ $$\int_\alpha^\beta \int_a^b f(R\cos\theta, R\sin\theta)\, R\, dR\, d\theta$$

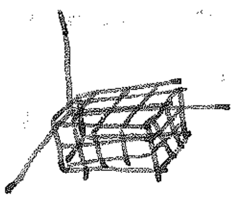
$D = \{(R,\theta) \mid \alpha \le \theta \le \beta,\ a \le R \le b\}$

general domain $\longrightarrow$ $$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(R\cos\theta, R\sin\theta)\, R\, dR\, d\theta$$

$D = \{(R,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le R \le h_2(\theta)\}$

LEC 15 - Center of mass, triple $\int$

- Center of mass : mass: $m = \iiint_E \rho(x,y,z)\, dV$, moments: $M_{oo} = \iiint_E o\, \rho(x,y,z)\, dV$ $(o = x \vee y \vee z)$, center: $\begin{cases} \bar{x} = \frac{M_{yz}}{m} \\ \bar{y} = \frac{M_{xz}}{m} \\ \bar{z} = \frac{M_{xy}}{m} \end{cases}$
- Triple Riemann sum :



$$\iiint_B f(x,y,z)\, dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

FUBINI $\longrightarrow$

$$\iiint_{B=[a,b]\times[c,d]\times[r,s]} f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

- $\iiint$ over general regions
  - Defined the same way as $\iint$
  - Type I : $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$ ($D \to$ proj xy-plane (E))

    $$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$
  - Type II : $E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$ ($D \to$ proj yz-plane (E))
  - Type III : $E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$ ($D \to$ proj xz-plane (E))

LEC 16 - Triple integral in cylindrical coordinates

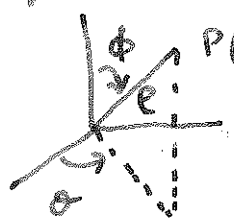
- Cylindrical = 3D polar

$$\begin{cases} R^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases} \qquad \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \qquad z = z$$

$$\iiint_E f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(R\cos\theta, R\sin\theta)}^{u_2(R\cos\theta, R\sin\theta)} f(R\cos\theta, R\sin\theta)\, R\, dz\, dR\, d\theta$$

LEC17 - Spherical coordinates, change of variables

· Spherical coordinates



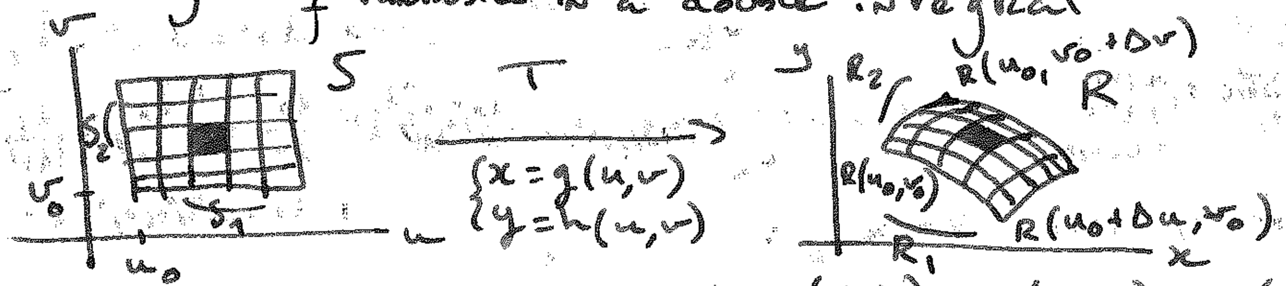
$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

cart 2D → polar
. Rectangle . polar rect.
cart 3D → cylindrical
. box . spherical wedge

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

E → spherical wedge $E = \{(\rho,\theta,\phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$

· Change of variables in a double integral



$(x,y) = R(u,v) = g(u,v)\,\vec{i} + h(u,v)\,\vec{j}$

→ $A_R = R_1 \times R_2$ $\left(\begin{array}{l} R_1 = R(u_0+\Delta u, v_0) - R(u_0,v_0) \\ R_1 \approx \Delta u\, R_u \end{array} \right)$ → because $R_u = \lim_{\Delta u \to 0} \frac{R(u_0+\Delta u, v_0)}{\Delta u}$

$\approx |\Delta u R_u \times \Delta v R_v|$

$A_R \approx \underbrace{|R_u \times R_v|}_{} \Delta u \Delta v$

→ Jacobian (vector): $R_u \times R_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \underbrace{\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}}_{\text{Jacobian}} \vec{k}$

$$\Rightarrow J_T = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

→ $A_R \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$

$\iint_R f\, dA_R = \iint_S f\, dA_S$

$$\iint_R f(x,y)\,dA \approx \sum\sum f(x_i,y_i)\,\overbrace{\Delta A}^{=A_R}$$

$$\approx \sum\sum f(g(u_i,v_i), h(u_i,v_i)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

Condition
→ T is one-to-one
→ J is nonzero

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

LEC18 - Vector fields, Line integrals in 2D

- Vector Field

$$F = P\vec{i} + Q\vec{j} \quad (2D) \quad \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$

$$F = P\vec{i} + Q\vec{j} + R\vec{k} \quad (3D) \quad \begin{pmatrix} P(x,y,z) \\ \ldots \end{pmatrix}$$

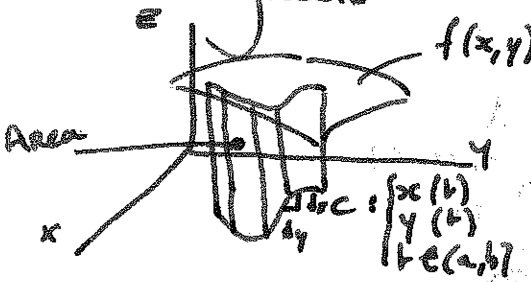
vectorfield $\to F : \mathbb{R}^2 \ (\mathbb{R}^3) \to V_2 \ (V_3)$

Scalar fields $\to Q(x,y)$, $P(x,y)$, $(R(x,y,z)\ldots)$

- Gradient vector field

$$\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$$

- Line integrals



$$\text{Area} = \int_C f(x,y)\,ds \to ds^2 = dx^2 + dy^2$$

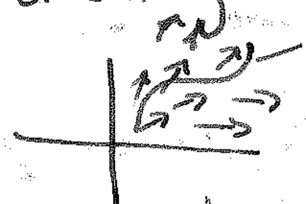
$$\text{Area} = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \quad \Bigg| \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$ds = \sqrt{(\ldots)^2+(\ldots)^2}\,dt$$

$\to$ line int w/ respect to x, y:

- $\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$ (same : dy)
- $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$

- Line integral of vector field



work of vector field on particle following path.

$$dW = \vec{f} \cdot d\vec{R} \to$$

$\vec{f}$, $d\vec{R}$, $\text{proj}_{d\vec{R}} \vec{f}$ $\Rightarrow$ dot prod.

$$\Rightarrow \int_C \vec{f} \cdot d\vec{R}$$

$$\int_C \vec{f} \cdot d\vec{R} = \underbrace{\int_a^b \vec{f}(\vec{R}(t)) \cdot R'(t)\,dt}_{} = \int_C \vec{f} \cdot \vec{T}\,ds$$

$\Big\{$ $\to$ parametrization w/ respect to t: $\vec{f}(x(t), y(t), z(t)) = \vec{f}(\vec{R}(t))$, $R'(t)\,dt = \vec{R}'(t)$

$\to$ tangent vector: $\vec{T}(t) = \dfrac{R'(t)}{|R'(t)|}$

⚠ $\vec{R} : \mathbb{R} \to \mathbb{R}^2$

$t \mapsto (x(t), y(t)) = x(t)\vec{i} + y(t)\vec{j}$

$\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (P(x,y), Q(x,y)) = P(x,y)\vec{i} + Q(x,y)\vec{j}$

with $\vec{i} = (1,0)$, $\vec{j} = (0,1)$ in $\mathbb{R}^2$ vector space (2D)

LEC19 - Line integrals in 3D

- Fundam^tl th^m for line integrals

$$\int_C \underbrace{\nabla f}_{\substack{\text{scalar} \\ \text{function} \\ \to \text{potential}}} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof: $\int_C \nabla f(\vec{r}(t)) \cdot r'(t)\,dt$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

- Path independence

$\int_C \vec{F} \cdot d\vec{r}$ is path ind. in D $\Leftrightarrow$ $\vec{F}$ is conservative

$\Leftrightarrow \exists f,\ \vec{F} = \nabla f$

$\Leftrightarrow \forall$ closed path $\in D$, $\int_{\text{closed path}} \vec{F} \cdot d\vec{r} = 0$

$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is conservative

$(\Leftarrow)\Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ *

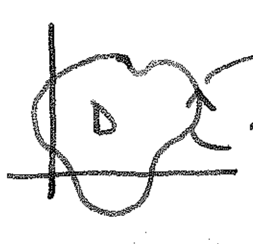
- Simply connected region D

$\forall$ simple closed curve $\in D$, pts enclosed are in D.

i.e.

*
$\left\{\begin{array}{l} \vec{F} = P\vec{i} + Q\vec{j} \text{ on sim-con region } D \\ P \wedge Q \text{ have cont. } 1^{st} \text{ order deriv.} \\ \text{Throughout } D,\ \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \end{array}\right. \Rightarrow \vec{F}$ is conservative

LEC 20 - Green's Theorem



positive orientation

$\partial D = C$ — positively (CCW) oriented boundary

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

2D counterpart of FTC

$F(b) - F(a) = \int_a^b F'\,dx$

in 1D, boundary is just 2 pts

LEC 21 - Divergence, curl, parametric surface

- vector differential operator : del (∇) ↳ NABLA ^_^

scalar → grad $f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ ↳ $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \ldots$

dot → div $\vec{f} = \nabla \cdot \vec{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

CROSS → curl $\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

- Laplace operator : laplacian (Δ)

$$\Delta = \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (= \text{div grad})$$

- Resulting theorems
  - → $f(x,y,z)$ has cont. 2nd order partial deriv. ⇒ curl grad $f = 0$
  - → $\vec{F}(x,y,z)$ has components whose 2nd o.p.d. are cont.
    - ⇒ curl $\vec{F} = 0$ ⇔ $\vec{F}$ is conservative
    - ⇒ div curl $\vec{F} = 0$

- Notes (optional!)
  - 2nd order derivatives: $\begin{cases} \nabla(\nabla\cdot\vec{F}) : \text{grad div } \vec{F} \\ \nabla\cdot\nabla f : \text{div grad } f \\ \nabla\cdot\nabla\times\vec{F} : \text{div curl } \vec{F} \\ \nabla\times\nabla f : \text{curl grad } f \\ \nabla\times\nabla\times\vec{F} : \text{curl curl } \vec{F} \end{cases}$
  - vector laplacian ∇^2

$$\nabla^2\vec{F} = \nabla(\nabla\cdot\vec{F}) - \nabla\times(\nabla\times\vec{F})$$
$$= \text{grad div } \vec{F} - \text{curl curl } \vec{F}$$

  - tensors, tensor fields, tensor product

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}$$

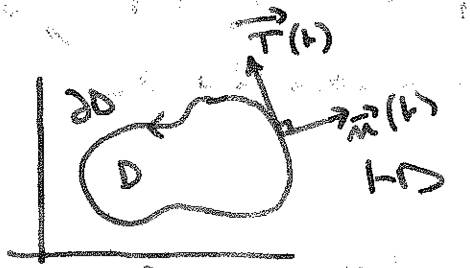
Note: $\nabla(\nabla\cdot\vec{F}) = \nabla\cdot(\nabla\otimes\vec{F})$

- Vector forms of Green

$$\oint_{\partial D} \vec{F}\cdot d\vec{R} = \oint_{\partial D} P\,dx + Q\,dy$$

$$\mapsto \operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\Rightarrow \oint_{\partial D} \vec{F}\cdot d\vec{R} = \iint_D (\operatorname{curl}\vec{F})\cdot\vec{k}\,dA$$



$\|\vec{T}(t)\| = \|\vec{n}(t)\| = 1$

$$\mapsto \oint_{\partial D} \vec{F}\cdot\vec{n}\,ds = \int_a^b (\vec{F}\cdot\vec{n})(t)\,|r'(t)|\,dt$$

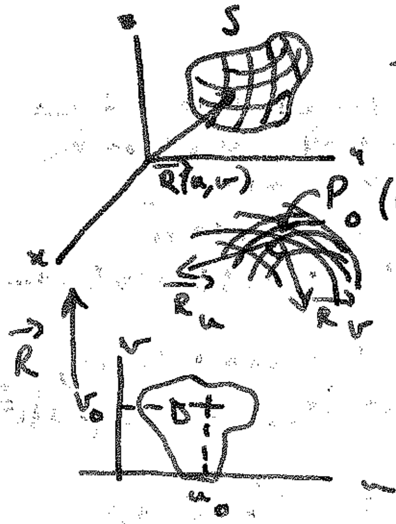
$$= \int_a^b P(x,y)\,y'(t) - Q(x,y)\,x'(t)\,dt$$

$$\oint_{\partial D} \vec{F}\cdot\vec{n}\,ds = \oint_{\partial D} P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \quad (1)$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q \quad (2)$$

$$\overset{(1)\wedge(2)}{\Longrightarrow} \oint_{\partial D} \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}\vec{F}(x,y)\,dA$$

- Parametric Surfaces



$$\vec{R}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

$P_0(u_0,v_0)$ $\quad S : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ $\quad \left| \begin{array}{l} \vec{R}_u = \frac{\partial}{\partial u}x\,\vec{i} + \frac{\partial}{\partial u}y\,\vec{j} + \frac{\partial}{\partial u}z\,\vec{k} \\ \vec{R}_v = \text{similar} \end{array}\right.$

$\forall (u,v)\in D,\ \vec{R}_u(u,v)\times\vec{R}_v(u,v) = 0 \Rightarrow S$ is <u>smooth</u>

(if $\vec{R}_u \parallel \vec{R}_v$, then corners!)

- Surface area

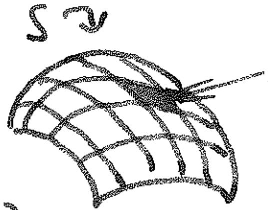
$\begin{cases} S \text{ is smooth} \\ S \text{ is traversed once when } (u,v) \text{ describe } D \end{cases} \Rightarrow A(S) = \iint_D |R_u \times R_v|\,dA$

- Special case : graph of func°

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \quad \left(\begin{array}{l} R_x = \vec{i} + \frac{\partial f}{\partial x}\vec{k} \\ R_y = \vec{j} + \frac{\partial f}{\partial y}\vec{k} \\ \text{then normalize x-prod} \end{array}\right)$$

LEC 22 - Surface integral

- For scalar fields



surface element: $|\vec{R}_u \times \vec{R}_v|$ → f is a scalar field defined over the surface S ⟶

$(u,v) \in D$ $\Longrightarrow \iint_D f(\vec{R}(u,v))\,|\vec{R}_u \times \vec{R}_v|\,dA$ S

$\vec{R}(u,v)$ maps cartesian coordinates to <u>curvilinear coordinates</u>

<u>Note</u>: $f(x,y,z) = 1 \Rightarrow \iint_D |\vec{R}_u \times \vec{R}_v|\,dA = \text{Area}$

- Sepcial case: S is graph of function $z = g(x,y)$

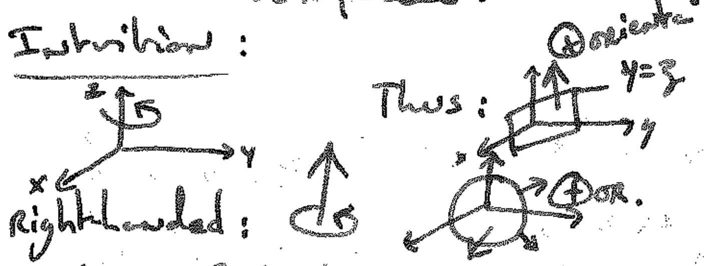
$\Rightarrow$ parametrization is given by: $\begin{cases} x = x \\ y = y \\ z = g(x,y) \end{cases}$

$$\Rightarrow \iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\,|\vec{R}_x \times \vec{R}_y|\,dA$$

<u>with</u>: $|\vec{R}_x \times \vec{R}_y| = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k} \Rightarrow |\vec{R}_x \times \vec{R}_u| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$

- Oriented surfaces:

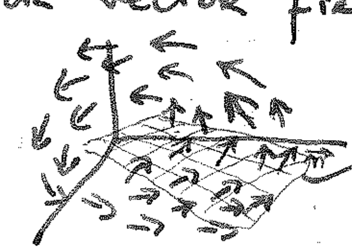
<u>Intuition</u>:



<u>Definition</u>

- orientable surface: two-sided ie NOT Möbius strip / klein bottle
- orientation of $\vec{n}$ (NORMAL vector) $= \frac{R_u \times R_v}{|R_u \times R_v|}$ for param rep. of suf. $R(u,v) = x(u,v)\vec{i} + \ldots$

- For vector fields



'path': surface S with normal (oriented) vector (oriented) S: $\vec{n}(u,v)$

Rate of flow through S, subject to (velocity) vector field $\vec{F}$

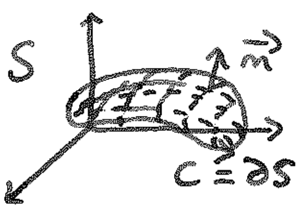
$$\Longrightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\,dS$$

(Surface integral of S is surface integral of its normal component)

For $\vec{n}(u,v) = \frac{\vec{R}_u \times \vec{R}_v}{|\vec{R}_u \times \vec{R}_v|}$, since $dS = |\vec{R}_u \times \vec{R}_v|\,dA$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\,dS = \iint_D \vec{F}(\vec{R}(u,v)) \cdot (\vec{R}_u \times \vec{R}_v)\,dA$$

LEC 23 - Stoke's Theorem



positive orienta° of the boundary curve: if the normal vector $\vec{n}$ goes around it in ⊕ orienta°, the surface S will always be on its <u>left</u>

$$\oint_{\partial S} \vec{F} \cdot d\vec{R} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$$

$$\oint_{\partial S} \vec{F} \cdot d\vec{R} = \oint_{\partial S} \vec{F} \cdot \vec{T}\, ds \ \wedge \ \iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \iint_S \text{curl}\, \vec{F} \cdot \vec{n}\, dS \Rightarrow$$

"line integral over ∂S of tangential comp. of $\vec{F}$ is equal to surface integral over S of normal comp. of curl $\vec{F}$"

LEC 24 - The Divergence Theorem

GREEN

$$\oint_{\partial D} \vec{F} \cdot \vec{n}\, ds = \iint_D \text{div}\, \vec{F}(x,y)\, dA \longmapsto$$

DIVERGENCE

$$\oiint_{\partial E} \vec{F} \cdot \vec{n}\, dS = \iiint_E \text{div}\, \vec{F}(x,y,z)\, dV$$

- conditions:
  - $\vec{S}$ ($\sim \partial E$) is the positively oriented closed boundary surface of a simple solid region E.
  - $\vec{F}$ has comp. with cont. partial deriv. on domain containing E

$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\, \vec{F}\, dV$$

- Fundamental Theorem of Calculus

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

- Gradient Theorem (FTC for line int)

$$\oint_C \nabla f \cdot d\vec{R} = f(\vec{R}(b)) - f(\vec{R}(a))$$

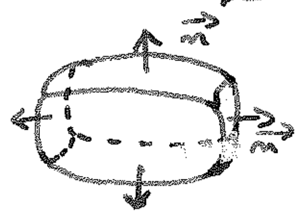
- Green's Theorem

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_{\partial D} P\, dx + Q\, dy$$

- Stoke's Theorem

$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{R}$$

- Divergence Theorem



$$\iiint_E \text{div}\, \vec{F}\, dV = \oiint_{\partial E} \vec{F} \cdot d\vec{S}$$