LEC01 - Linear Systems, Matrix Algebra

→ System of linear equations

$$S = \begin{cases} \sum_{i=1}^{n} a_i x_i = \alpha \\ \dots\dots \text{ m lines} \end{cases}$$

S is <u>consistent</u> iff S admits at least 1 solu°
S is <u>inconsistent</u> iff S admits no solu°
$\alpha, \beta \dots = 0 \Rightarrow$ S is <u>homogeneous</u>

- Systems are equivalent if they have the same <u>solution set</u>.

$$S \Leftrightarrow Ax = b \left(\Leftrightarrow [A|b]\right)$$

- Matrix representation:

$$\underbrace{\begin{bmatrix} a_{11} \dots a_{1n} \\ (\dots) \\ a_{m1} \dots a_{mn} \end{bmatrix}}_{\text{coefficient matrix}}^{A} \underbrace{\begin{bmatrix} x_1 \\ (\dots) \\ x_n \end{bmatrix}}_{\text{n-vector}}^{x} = \overset{b}{\begin{bmatrix} \alpha \\ (\dots) \\ \end{bmatrix}} \quad \Big| \quad \underbrace{\begin{bmatrix} a_{11} \dots a_{1n} & \alpha \\ (\dots) & (\dots) \end{bmatrix}}_{\text{augmented matrix}}^{[A|b]}$$

$\Updownarrow$

- linear combination of column vectors: $x_1 \begin{bmatrix} a_{11} \\ (\dots) \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ (\dots \\ a_{m2} \end{bmatrix} \dots = \begin{bmatrix} \alpha \\ (\dots) \\ \end{bmatrix}$

→ Matrix algebra

- addition: commutative, associative, neutral element: <u>zero matrix</u> $[0]$
- scalar product: commutative, associative, distributive (+)
- transposition:
  - → $A^T = [a^T_{ij}]$ is the transpose of $A = [a_{ij}]$ iff $\begin{cases} A_{m\times n} \Rightarrow A^T_{n\times m} \\ \forall_{i,j}, a^T_{ij} = a_{ji} \end{cases}$
  - → $(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $cA^T = (cA)^T$, $\underline{\underline{(AB)^T = B^T A^T}}$
- Multiplication:
  - → AB is the product of $A_{m\times n}$ and $B_{n\times p}$ iff $\begin{cases} AB_{m\times p} = [c_{ij}] \\ \forall_{ij}, c_{ij} = \sum_{i=1}^{p} a_{ik} b_{kj} \end{cases}$
  - → non-commutative, neutral element: <u>identity matrix</u> I_n
  - → Resulting matrix hosts dot product of rows of A and cols of B
- exponentiation: $A^0 = I_n$, $A^p A^q = A^{p+q}$, $(A^p)^q = A^{pq}$, $(AB)^p = A^p B^p \Leftrightarrow AB = BA$

→ Square Matrices

- <u>invertible</u> (= nonsingular) iff $\exists! B$, $BA = AB = I_n$ (≠ <u>singular</u> = noninvertible)
  - → $AA^{-1} = A^{-1}A = I_n$, $(A^{-1})^{-1} = A$, $(A^T)^{-1} = (A^{-1})^T$
  - → $(A_1 \times A_2 \dots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \dots A_1^{-1}$
- <u>square matrix</u>: A is $n \times n$
  - <u>diagonal matrix</u>: square $\wedge \forall i,j,\ i \neq j \Rightarrow a_{ij} = 0$
    - <u>scalar matrix</u>: diagonal $\wedge \exists! \alpha, \forall_{i,j},\ i = j \Rightarrow a_{ij} = \alpha$
      - <u>identity matrix</u>: scalar $\wedge \forall_{i,j},\ i = j \Rightarrow a_{ij} = 1$
- <u>upper triangular matrix</u> (resp. <u>lower triangular matrix</u>): $\begin{bmatrix} x & x & x \\ o & x & x \\ o & o & x \end{bmatrix} \left(\begin{bmatrix} x & o & o \\ x & x & o \\ x & x & x \end{bmatrix}\right)$
- <u>symmetric matrix</u>: $A^T = A$ (i.e. $\forall_{i,j}, a_{ij} = a_{ji}$)
- <u>skew symmetric matrix</u> $A^T = -A$ (i.e. $\forall_{i,j}, a_{ij} = -a_{ji}$)

LEC02 - Solving Systems of linear equations

→ Echelon Form

- Row echelon form (REF) → all rows of zeros are at bottom of matrix
  - → first nonzero entry of every row is a leading one
  - → leading one are to the right and below one another
- Reduced row echelon f. (RREF) → every column contains a single leading one.
- Column echelon form and reduced CEF are transpositions of those.

⚠ Every square matrix in RREF is either I_m or has a row of zeros $[0 \ldots 0]$

→ Elementary operations

- elementary row (col.) operations → interchange two rows: $R_i \leftrightarrow R_j$ (or cols)
  - → Multiply row (or col.) by nonzero num: $R_i \leftarrow kR_i$
  - → linear combinations of rows (or cols): $R_j \leftarrow kR_i + R_j$
- Matrices are row (col.) equivalent if they differ by a sequence of elem. row (col.) ops.
- $\forall$ nonzero matrices, $\exists$ equiv matrix in REF ; $\exists!$ equiv matrix in RREF.
- If the augmented matrices of two linear systems are row equivalent, the systems are equiv.

→ Gaussian elimination

- Process to reduce matrix to REF
  - → forward elimination: use elementary row ops to obtain REF (or degenerate)
  - → backward substitution: substitute results bottom-up
- Gauss-Jordan elimination: use elementary row ops to obtain RREF (or degen.)
- linear system has
  - 0 solutions if aug matrix becomes degenerate : $[0\ 0\ 0 \ldots 1 | x]$
  - 1 solution if aug matrix becomes identity : I_m $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
  - ∞ solutions if aug matrix has rows of zeros: $[0\ 0\ 0 \ldots 1\ 0]$

→ Elementary matrices

- An elementary matrix is a matrix that differs from I_m by a single elementary row op.
- Performing an elem. row op on A is equivalent to premultiplying A by the elementary mat. that differs from I_m by the same operation : $B = A_{op} \Leftrightarrow B = E_{op} A$
- A and B are row-equiv. iff $\exists (E_i)$, $B = E_k \cdots E_1 A$ (with $E_1 \ldots E_k$ elem. mat.)

→ Nonsingular matrices

- Equivalent statements : A is invertible $\Leftrightarrow$ A is the product of elementary matrices
  - $\Leftrightarrow$ A is row-equiv to I_m $\Leftrightarrow$ $A\vec{x} = \vec{0}$ has a unique trivial solu°
  - $\Leftrightarrow$ $A\vec{x} = \vec{b}$ has a unique solution (per vector $\vec{b}$)
- Finding the inverse: apply Gauss-Jordan on augmented matrix with I_m: $[A | I_m]$ to obtain : $[I_m | A^{-1}]$, then: $E_k \ldots E_1 [A | I_m] = [I_m | A^{-1}]$

⚠ the inverse of an elem. matrix is obtained easily by performing the exact opposite op. to I_m

$$\Longrightarrow \begin{cases} A^{-1} = E_k \cdots E_1 \\ A = [E_k \cdots E_1]^{-1} = E_1^{-1} \cdots E_k^{-1} \end{cases}$$

LEC 03 - Vector spaces and subspaces

→ Real vector spaces

- Def: set V with two operations: . vector addition $+$: $V \times V \to V$, such that:
  (1) identity elem $(a+0=a)$ / (2) associativity $(a+(b+c))$ / (3) commutativity / (4) negative $(a-a=0)$
  - scalar multiplication $*$: $V \times \mathbb{R} \to V$ such that:
  (5) identity elem $(1*v=v)$ / (6) associativity / (7) dist. ~vec+ $(k(\vec{a}+\vec{b}))$ / (8) dist. ~field + $((k+l)\vec{a})$

→ Subspaces

- Def: subset of vectorspace V with same properties on $+$ and $*$.
- W is a subspace of V iff $W \subseteq V \land W$ is <u>closed</u> under $+$ and $*$.

⚠ $\forall$ vecspc V, V and $\{0\}$ are subspaces of V.

LEC 04 - Span & Linear Independence

→ Span

- Def: set of linear combinations: $\text{span}\{V_1 \ldots V_m\} = \{w \mid w = \sum_{i=1}^{m} a_i v_i\}$
- $S \subseteq V \Rightarrow \text{span}(S)$ is a subspace of V.
- $\text{span}(S) = V \iff S$ is a <u>spanning set</u> of V.

→ Linear Independence

- Def: set of vectors $\{V_1 \ldots v_m\}$ is <u>linearly independent</u> iff $\sum a_i v_i = 0 \iff \forall a_i, a_i =$
  i.e. no vector in $\{v_1 \ldots v_m\}$ is a linear combination of others.
- To show that property, consider the coefficient matrix $V = [V_1 \ldots V_m]$. Use Gauss-Jordan to show that $V\vec{x} = \vec{0}$ admits a unique trivial solution, if not, the vectors are <u>linearly dependent</u>.

LEC 05 - Basis, Dimension, Rank

→ Basis

- Def: $\{v_1 \ldots v_m\}$ is a <u>basis</u> for vecspc V iff $\begin{cases} \text{span}\{V_1 \ldots V_m\} = V \\ \{V_1 \ldots V_m\} \text{ is linearly independent} \end{cases}$
  - if vectors in $\{V_1 \ldots V_m\}$ are <u>pairwise orthogonal</u>, $\{v_1 \ldots v_m\}$ form a <u>natural basis</u>.
    ↳ and for every v_i, all its values are 0 except for one (which is equal to 1)

→ Dimension

- Def: $\dim V$ (:: vecspc) $= \begin{cases} 0 & \text{if } V = \{0\} \\ |B|, \text{ where } B \text{ is a basis for } V & \text{otherwise} \end{cases}$
- V is <u>finite-dimensional</u> iff $\exists$ finite subset of V such that it is a basis for V.
- The <u>maximal independent subset</u> T of S is a subset of S such that T is linearly indep. and there is no other such subset in S that properly contains T.
- Thm: $S \subseteq V \land \text{span } S = V \Rightarrow$ Max indep. subs of S is a basis for V.

⚠ S is a basis for $V \iff \exists!$ lin. combi of vecs in S equal to a given vector in V

- Corollaries: → $\dim V = n \Rightarrow \forall S \subseteq V, |S| > n \Rightarrow S$ is lin. dep.
  - $\Rightarrow \forall S \subseteq V, |S| < n \Rightarrow \operatorname{span} S \neq V$
  - $\Rightarrow \forall S \subseteq V, |S| = n \wedge \operatorname{span}(S) = V \Rightarrow S$ is a basis for V
  - $\Rightarrow \forall S \subseteq V, |S| = n \wedge S$ is lin. indep. $\Rightarrow S$ is a basis for V
- Process for finding a basis for V:
  - → Find $S \subseteq V$, such that $\operatorname{span} S = V$ $(S = \{v_1 \dots v_n\})$
  - → $\exists T \subseteq V$, such that T is a basis for V (T is the max indep. subs. of S)
  - → Use Gaussian elimination on augmented matrix of homogeneous system $[v_1 \dots v_n | \vec{0}]$
  - → Rows with a leading one are those in T.

→ Rank

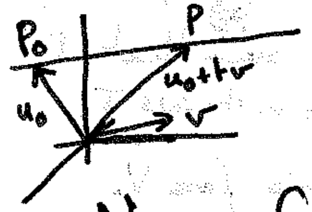
- The <u>column space of A</u> (:: matrix) is the span of the col vectors of $A (\in \mathbb{R}^n)$
- The <u>row space of A</u> (:: matrix) is the span of the row vectors of $A (\in \mathbb{R}_n)$
- <u>Row rank</u> (A) = dim (rowspc A) ; <u>col rank</u> (A) = dim (colspc A)
- <u>Thm:</u> A row-equiv B ⇒ rowspc (resp. colspc) A = rowspc (resp. colspc) B
- Process for finding a basis for V:
  - → repeat previous method by forming $\underbrace{\text{aug. mat. of hom. sys}}_{\text{where span rowspc } A = V}$ $[v_1 \dots v_n | 0]$
  - → nonzero rows will form basis for rowspc of REF(A). Since A ⇔ REF(A) ⇒ colspc A ⇔ colspc REF(A), it is also a basis for rowspc A.
  - Note: we can apply this method to find a basis for the col space of A by using A^T. This would be equiv. to using A with CEF(A).
- Rank A = row rank A = column rank A (⇔ Rank A = Rank A^T)
  - Note: Rank A is the number $\begin{cases} \text{of pivots in REF}(A) \\ \Leftrightarrow \text{nonzero rows in REF}(A) \text{ or REF}(A^T) \end{cases}$

→ Nullity

- The <u>Null Space</u> (or <u>Kernel</u>) of a matrix $\overset{A}{\vee}$ is the solution space of $A\vec{x} = \vec{0}$.
- <u>nullity (A)</u> = dim (null spc A)
- <u>Thm</u> (rank-nullity): $A_{m \times n} \Rightarrow \begin{cases} \text{Rank } A + \text{nullity } A = n \text{ (nb of cols in } A) \\ \text{Rank } A + \text{nullity } A^T = m \text{ (nb of rows in } A) \end{cases}$

LEC06 - Vector Geometry

→ Lines in $\mathbb{R}^3$



$P_0 = (x_0, y_0, z_0) \Rightarrow \vec{u}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

$P = (x, y, z) \Rightarrow \underbrace{\vec{u} = \vec{u}_0 + t\vec{v}}_{\text{parametric equation}}$

- Norm of v in Eucl. Space: $\|v\| = \sqrt{v_1^2 + \dots v_n^2}$

- distance in Eucl. Sp: $d(A,B) = \|\vec{AB}\| = \|\vec{OB} - \vec{OA}\|$ (where $\vec{OX} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$)

→ Geometric interpretation of subspaces of $\mathbb{R}^n$

- subspaces of $\mathbb{R}$ are: $\{0\}$ and $\mathbb{R}$.
- subspaces of $\mathbb{R}^2$ are: $\{0\}$, $\mathbb{R}^2$ and any line passing through the origin.
- subspaces of $\mathbb{R}^3$ are: $\{0\}$, $\mathbb{R}^3$ and
  any line ———— origin,
  any plane ———— origin.

→ Dot Product

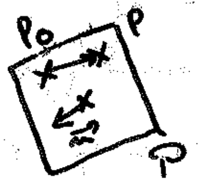
$\forall u, v \in \mathbb{R}^n,\ u \cdot v = u^T v$

$\left(\to u, v \in \mathbb{R}^2,\ [u_1\ u_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2 \right)$

- $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$
- $\vec{u} \cdot \vec{u} \geq 0 \quad (\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0})$
- $\mathrm{comp}_{\vec{u}}(\vec{v}) = (\vec{u} \cdot \vec{v}) \Rightarrow \mathrm{proj}_{\vec{u}}(\vec{v}) = (\vec{u} \cdot \vec{v}) \dfrac{\vec{u}}{\vec{u} \cdot \vec{u}}$

$\left(\mathrm{proj}_u(\vec{v}) = \mathrm{comp}_u(\vec{v}) \times \frac{\vec{u}}{\|\vec{u}\|^2} \right)$

→ Planes in $\mathbb{R}^3$



$P \in \mathcal{P} \iff \vec{n} \cdot \overrightarrow{P_0 P} = 0$

$\iff [n_1\, n_2\, n_3] \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$

$\iff n_1 (x - x_0) + \ldots = 0$

$\iff n_1 x + n_2 y + n_3 z + d = 0$

(with $d = -n_1 x_0 - n_2 y_0 - n_3 z_0$)

$\Rightarrow$ from parametric equ° $ax + by + cz + d = 0$, we can deduce: $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Note: unit vector in direc° of $\vec{u}$ is $\dfrac{\vec{u}}{\|\vec{u}\|}$.

Note: $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

LEC07 – Inner Product Spaces

→ Inner product spaces

- Def: Vector Space V with inner product •, such that: $\begin{cases} u \cdot u \geq 0 \land (u \cdot u = 0 \iff u = \vec{0}) \\ c(u \cdot v) = (cu \cdot v) = (u \cdot cv) \\ \bullet \text{ is dist over vec+} \\ \bullet \text{ is commutative} \end{cases}$

  (→ the inner product on $\mathbb{R}^n$ ($n \in \mathbb{N}$) is the dot product)
- A <u>Euclidian Space</u> is a finite-dimensional inner product space.
- <u>Cauchy-Schwartz Inequality</u>: $\forall u, v \in V (:: IPS),\ |u \cdot v| \leq \|u\| \times \|v\|$
- $\{v_1 \ldots v_n\} \subseteq V (:: IPS)$ is <u>orthogonal</u> iff $\forall u, v \in \{v_1 \ldots v_n\},\ u \perp v$ (equivalently: $u \cdot v = 0$)
  - $\{v_1 \ldots v_n\}$ is <u>orthonormal</u> iff $\{v_1 \ldots v_n\}$ is orthogonal and $\forall v \in \{u, v\},\ \|v\| = 1$ (i.e v is a unit vector)
- Thm: S is orthogonal ⇒ S is lin. indep.

→ Gram-Schmidt process

- $\forall V$ (:: Eucl. spc), $\exists S$ (:: orthogonal basis for V), and (let $\dim V = n$ and $S = \{u_1 \ldots u_n\}$):

  $\forall v \in V,\ v = \sum_{i=1}^{n} c_i u_i,\quad c_i = (v \cdot u_i)$ (i.e. every v in V can be written uniquely as lin. comb. of vecs in S where coefs are the $v \cdot u_i$)
- <u>Gram-Schmidt process</u> provides a constructive proof for:

  $\forall W$ (:: subspace of V (:: IPS)), $W \neq \{0\} \Rightarrow \exists T$ (:: orthonormal basis for W)

  → let S (:: basis for W) $= \{u_1 \ldots u_n\}$. Let $v_1 = u_1$.

  → Call $W_1 = \mathrm{span}(v_1, u_2)$. Let $v_2 \in W_1$ such that $\begin{cases} v_1 \cdot v_2 = 0 \\ v_2 = c_1 v_1 + c_2 u_2\ (c_2 \neq 0) \end{cases}$

  → Then $(c_1 v_1 + 1 u_2) \cdot v_1 = 0 \Rightarrow c_1 = -\dfrac{u_2 \cdot v_1}{v_1 \cdot v_1}$

  $\Rightarrow v_2 = u_2 - \dfrac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$

  → compute v_2 and deduce $W_1 = \mathrm{span}(v_1, v_2)$.

  → repeat the previous steps by defining everything: $v_3 = u_3 - \widetilde{\mathrm{proj}}_{v_1} u_3 - \widetilde{\mathrm{proj}}_{v_2} u_3$ (hence: constructing v_i by removing from u_i its projection on the other vectors).

  → not exactly proj because $u \cdot u = \|u\|^2$

  → Normalize the resulting basis.

→ Orthogonal Complement

pronounce: W perp

- Def: For any subspace W of an IPS V, the <u>ortho. comp.</u>: $W^{\perp} = \{v \in V \mid \forall w \in W, v \perp w\}$
- Thm: . $\forall W$ (:: subspace of IPS), $W^{\perp}$ is a subspace of same IPS and $W \cap W^{\perp} = \{0\}$
  - . $\forall W$ (:: subspace of V (:: IPS)), $\begin{cases} (W^{\perp})^{\perp} = W \\ \underline{W \oplus W^{\perp} = V} \end{cases}$

$\oplus$: direct sum. that is $\forall v \in V, \exists! (w_1, w_2) \in W \times W^{\perp}, v = w_1 + w_2$

$\exists! w_1 \in W, \exists! w_2 \in W^{\perp}$

→ Fundamental subspaces of a Matrix.

- $\forall A \in \mathbb{R}^{m \times n}$ $(A_{m \times n})$, A induces 4 fundamental subspaces: Null A, rowspc A, colspc A, Null A^T.

The <u>fundamental theorem of Linear Algebra</u> states that:

$$\begin{cases} \text{Row spc } A \subset \mathbb{R}^n, \text{ null } A \subset \mathbb{R}^n, \text{ rowspc } A = (\text{Null } A)^{\perp} \\ \text{col spc } A \subset \mathbb{R}^m, \text{ null } A^T \subset \mathbb{R}^m, \text{ colspc } A = (\text{null}(A^T))^{\perp} \end{cases} \left(\begin{array}{l} \Rightarrow \text{Rowspc } A \oplus \text{null} A = \mathbb{R}^n \\ \Rightarrow \text{colspc } A \oplus \text{null} A^T = \mathbb{R}^m \end{array} \right)$$

LEC08 - Linear Transformations

→ Linear transformations

$\begin{cases} \text{CALC I: linear = affine} \\ \text{LinAlg: linear = affine through origin} \end{cases}$

- Def: $L: V$ (:: vec spc) $\rightarrow W$ (:: vecspc), such that:
  - . $L(\vec{u} + \vec{v}) = L(\vec{u}) + L(\vec{v})$ $(\forall \vec{u}, \vec{v} \in V)$
  - . $L(c\vec{u}) = cL(\vec{u})$ $(\forall \vec{u} \in V, \forall c \in \mathbb{R})$

[L is a linear map from V to W.

[L is a lin. tr. of V into W.

[L is a lin. operator on V.

- Direct consequences:
  - . $L(u - v) = L(u) - L(v)$ $(\forall \vec{u}, \vec{v} \in V)$
  - . $\underline{L(0_V) = 0_W}$
- Thm: Every matrix transformation ($[u \mapsto Au]$) is a lin. trans.
- Thm: $L: V \rightarrow W \wedge L': V \rightarrow W$. L, L' (:: lin. trans.). $S = \{u_1 .. u_n\}$ (:: basis for V)

$$[\forall w \in S, L(w) = L'(w)] \Rightarrow \forall v \in V, L(v) = L'(v)$$

- Thm: If V, W are finite dimensional, every lin. trans can be repr. by a matrix trans.

(special case). $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$. L (:: lin trans). $\{e_1 .. e_n\}$ is <u>the natural basis</u> of $\mathbb{R}^n$.

$$\forall x \in \mathbb{R}^n, L(x) = Ax \quad (\text{where } A = [L(e_1) \dots L(e_n)])$$

note: A is unique, it is the standard matrix representing L.

(general case) . $L: V \rightarrow W$. L (:: lin trans). $\dim V = n$, $\dim W = m$ $(n, m \neq 0)$.

$$\forall (T, S) \in V \times W, T (:: \underline{\text{ordered basis}} \text{ for } V), S (:: \text{or.b. for } W), \exists! A, \forall v \in V, \underline{\underline{[L(v)]_T = A[v]_S}}$$

<u>Note</u>: basis where order of vectors is fixed. →

<u>note</u>: $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ (where $v = a_1 w_1 + \dots a_n w_n$ where $S = \{w_1 .. w_n\}$ is an ordered basis.) is the coordinate vector of v with respect to the ordered basis S.

SORRY for mess!

—▷ Kernel

- Def: For any lin. trans. L, the <u>kernel</u> of L: $\ker L = \{v \in V \mid L(v) = 0_W\}$
  - note: $\forall L$ (::lin trans), $\ker L \ni \{0_V\}$
- Thm:
  - $\ker L$ (:: subspace of V)
  - L is one-to-one iff $\ker L = \{0_V\}$
  - $\forall x, y \in V,\ L(x) = L(y) \Rightarrow x - y \in \ker L$

—▷ Range

- Def: For any lin. trans. L, the <u>range</u> of L (or <u>image</u> of L): $\operatorname{im} L = \text{range } L = \boxed{\{w \in W \mid \exists v \in V, L(v) = w\}}$
- Thm:
  - $\operatorname{im} L$ (:: subspace of W)
  - L is onto iff $\operatorname{im} L = W$
- Thm: $\dim \ker L + \dim \operatorname{im} L = \dim V$
  - $[\dim V = \dim W] \Rightarrow [L \text{ is one-to-one} \Leftrightarrow L \text{ is onto}]$

—▷ Invertibility

- Def: L is <u>invertible</u> iff $\exists! L^{-1}: W \to V,\ L \circ L^{-1} = I_W \wedge L^{-1} \circ L = I_V$
- Thm: L is invertible iff L is one-to-one $\wedge$ L is onto

(• Thm: L is one-to-one iff $\forall S \subset V$, S (::lin. indep.) $\Rightarrow L(S)$ (::lin. indep.))

LEC09 - Determinants

permutation group of n — signature of permuta° (#switches)

- Def: for any $n \times n$ matrix A: $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma_i}$
  - e.g. $S_2 = \{12, 21\}$ $\Rightarrow \det(A_2) = +A_{11}A_{22} - A_{12}A_{21}$
- Thm: $\det(A) = \det(A^T)$
- Note: det is an alternating multilinear map, thus:
  - $k\det(R_1 \ldots R_n) = \det(kR_1 \ldots R_n)$
  - $\det(R_1 \ldots R_i, R_j \ldots R_n) = \det(R_1 \ldots R_i + R_j, R_j \ldots R_n)$
  - $\det(R_1 \ldots R_i, R_j \ldots R_n) = -\det(R_1 \ldots R_j, R_i \ldots R_n)$
  - $R_i \leftarrow kR_i \Rightarrow \det \to k\det$
  - $R_i \leftarrow R_i + kR_j \Rightarrow \det \to \det$
  - $R_i \leftrightarrow R_j \Rightarrow \det \to -\det$
- Thm: $A_{n\times n} \wedge B_{n\times n} \Rightarrow \det(AB) = \det(BA) = \det(A)\det(B)$
- Thm: $A_{n\times n}$ invertible $\Leftrightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$ $(\det A \neq 0)$
  - $A_{n\times n}$ singular $\Leftrightarrow \det(A) = 0$

—▷ Cofactor Expansion

- Def:
  - the minor M_{ij} of $A_{n\times n}$ for entry a_{ij} is obtained from A by removing its ith row and jth col. and taking the resulting determinant.
  - the cofactor C_{ij} of $A_{n\times n}$ for entry a_{ij} is: $C_{ij} = (-1)^{i+j} M_{ij}$ $\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$
- Thm: $\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$ for any column j.
  - $= \sum_{j=1}^{n} a_{ij} C_{ij}$ for any row i.

CALC II → Jacobian:

$$\vec{J} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = +\vec{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\vec{k}$$

- Note: To manually compute determinants: reduce (keep track of $\det \to x\det$) then expand.

⚠ • Note: For a matrix in <u>REF</u>, the determinant will be the product along the diagonal! It follows that $\det A = 0$ iff A has a row of 0s iff A is singular.

→ Adjugate Matrix

- Def:
  - the cofactor matrix of $A_{m\times m}$ is $C_A = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{mm} \end{bmatrix}$ (matrix of its cofactors)
  - the adjugate (= adjunct) matrix of $A_{m\times m}$ is $\operatorname{Adj} A = C_A^T$ (transpose of)
- Thm: $A \times \operatorname{Adj} A = \operatorname{Adj} A \times A = \det(A)\, I_m$
  - (Cor: (a.k.a ⇒)) $A^{-1} = \frac{1}{\det(A)} \operatorname{Adj} A = \det(A^{-1}) \operatorname{Adj}(A)$

→ Cramer's Rule

- Def: procedure for solving systems with invertible $\overset{\text{square}}{\vee}$coef matrix $A_{m\times m}$.
  - → For $Ax=b$, since $x = A^{-1}b \wedge A^{-1} = \frac{\operatorname{Adj}(A)}{\det(A)} \Rightarrow x = \det(A)^{-1} \operatorname{Adj}(A)\, b$
  - → $\Rightarrow \forall i \in \{0..m\},\ x_i := \det(A)^{-1} \sum_{j=1}^{m} C_{ji} b_j$.
  - → And $\sum_{j=1}^{m} C_{ji} b_j$ is the cofactor expansion (along col. i) of A_i = A where col. i is replaced by col. vec. b
  - → $\Rightarrow \forall i \in \{0..m\},\ x_i = \frac{\det(A_i)}{\det(A)}$

LEC 10 - Eigenvalues & Eigenvectors

- Def: λ (::eigval $A_{m\times m}$) iff $\exists x \in \mathbb{R}^m \setminus \{0_m\},\ Ax = \lambda x$ # x (::eigvec A)

  Note: motivation from linear maps: eigenvectors of $T: V \to W,\ x \mapsto Ax$ are vectors in V that that their image by T does not change direction (only stretched by resp. eigenvalues)

→ Characteristic Polynomial

- Note: To find eigvals λ of A: $Ax = \lambda x \Leftrightarrow Ax - \lambda I_m x = 0_m \Leftrightarrow (A - \lambda I_m)x = 0_m$

  For $x \neq 0_m$, $(A - \lambda I_m)x = 0_m$ has (nontrivial) solutions iff $A - \lambda I_m$ is noninvertible iff $\det(A - \lambda I_m) = 0$
- Def: For $A_{m\times m}$, $\begin{cases} \det(A - \lambda I_m) \text{ is the characteristic polynomial of A (its degree is m)} \\ \det(A - \lambda I_m) = 0 \text{ is the characteristic equation of A} \end{cases}$
- Thm: The eigenvalues of $A_{m\times m}$ are the roots of its char poly (sols of char eq.)
- Def: The eigenspaces of $A_{m\times m}$ are the subspaces $E_\lambda = \operatorname{null}(A - \lambda I_m)$

  note: For each eigval λ, the corresponding E_λ will contain all respective eigvecs and the 0_m vec.

⚠ → Eigenvalues characterise matrices, so, in general, A and RREF(A) do not hold the same, → cf similarity

→ Diagonalization

⚠ $\dim E_\lambda \leq$ multiplicity λ ⚠

- Def: A_m is diagonalizable iff $\exists P_{m\times m}, D_{m\times m}$, P (::nonsingular), D (::diagonal), $P^{-1}AP = D$

  Note: $P^{-1}AP = D \Leftrightarrow AP = PD$. $\begin{cases} P = (c_1 .. c_m) \\ D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_m \end{pmatrix} \end{cases} \Rightarrow [Ac_1 \ldots Ac_m] = [\lambda_1 c_1 \ldots \lambda_m c_m] \Leftrightarrow \forall i \in \{1..m\},\ Ac_i = \lambda_i c_i$

  from this equa°: cols in P are eigvecs of A. Since P is invertible: cols in P are lin. indep. and basis $\mathbb{R}^m$
- Thm: A is diagonalizable iff A has m lin indep eigvecs that form a basis for $\mathbb{R}^m$

  Note: D is a diagonal matrix with eigenvalues of A along the diag. (there $\overset{\text{at most}}{\text{are}}$ m! possible D)

  ⚠ When $\dim E_\lambda$ = multiplicity $\lambda\ \forall \lambda$ (eigvals of A), A can be diagonalized. ⚠

LEC01 - Algebraic Structures

- Group : set and op $(G, \cdot)$ & $\cdot$ respects 4 axioms : assoc, closure, invertability, identity
  - └ Abelian grp : $\cdot$ respects commu
    - └ Ring : $(G, +, *)$ & $*$ respects distrib over $+$, assoc, identity
      - └ Commutative ring : $*$ respects commu (e.g. $\mathbb{Z}$)
        - └ Field : $*$ respects invertability (for nonzero) ~~[illegible] is nonzero~~ itself

Thus a field has 0, 1 distinguished. By closure we know that:

[nota: $\mathbb{F} = (F, +, *)$]

$\rightarrow 1 \in F$

$\rightarrow 1+1 \in F$

....

| GARY+COOK : terms $(1, 1+1, 1+1+1 \ldots)$ are (i.e. from axioms) (aximatically) syntactically constructed. Depending on cardinality of F, there will be "repeats" → terms having the same meaning

- Characteristic of a field : min # of times needed to add 1 to 0 to get a term with the same semantic as 0 OR 0 if it never reaches 0 (i.e. infinite field, e.g. $\mathbb{Q}$ is the smallest)

  ↳ $\mathbb{N}$: + not invertible , $\mathbb{Z}$: $*$ not invertible

  ↳ commutative monoid ↳ com. ring

  ↳ $(G, \cdot)$ & $\cdot$ is assoc, identity

- Galois fields (aka. finite fields)

  <u>Def</u>: field w/ nonzero characteristic $\Leftrightarrow |F| < \aleph_0$

  - underlying set contains term semantics (GARY: terms are endowed with meaning given by platonic world of the abstract) e.g. quantities 1,2,3...

  <u>Prop</u>: $(\Rightarrow)$ # of <u>elements</u> in a field is p^k, p prime, for some k

  $(\Leftarrow)$ For every prime p and natural k, $\mathbb{F}_{p^k}$ exists.

  - Two fields with same # of elements are isomorphic.

- Prime fields (i.e. $\mathbb{F}_{p^1}$ ← !)
  - $\mathbb{F}_2$ is the smallest field (by def of field : $0 \neq 1 \Rightarrow \min|F| = 2$), it is also a binary field (i.e. $p=2$ for some k).
  - Every prime field $\mathbb{F}_p$ is isomorphic to the ring of integers modulo ~~p~~ p: $\mathbb{Z}/p$.

Modular arithmetic redux

$a \equiv b \pmod{m} \Leftrightarrow a \equiv b\ [m] \Leftrightarrow (a-b) \text{ div } m = q \wedge (a-b) \bmod m = 0$, where $q \in \mathbb{Z}$

$a = b + qm \Leftrightarrow$

| where: div: $m, n \mapsto p$, p is the quotient of m ~~and~~ n

mod: $m, n \mapsto R$, R is the remainder of m / n

. Consider the func° $f_m : a \longmapsto [b]$ (i.e. $\forall a \in \mathbb{Z}$, $f_m(a) = a \bmod m$)

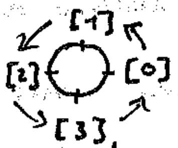
also written $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}_m$

MAT224 L2

$\mathbb{Z}/m = f_m(\mathbb{Z})$, i.e. $\mathbb{Z}$ (. . . . ∴) is mapped to $\mathbb{Z}/m$ (∴ ~~[illegible]~~ ≤ m dots)

. Congruence relation $\underbrace{\text{is an}}_{\text{modulo } m}$ equivalence relation ⇒ The ~~[illegible]~~ resulting equivalence class (~~aka~~: congruence class, residue class) is denoted: $[a] = \{a + km \mid k \in \mathbb{Z}\}$

. Thus $\mathbb{Z}/m$ is ~~called~~ a set of integers modulo m:

e.g. $\mathbb{Z}/4$ 

$0 \equiv 4 \equiv 8 \equiv \ldots \Leftrightarrow 0, 4, 8 \ldots \in [0]$

$\mathbb{Z}/4 = \{[0], [1], [2], [3]\}$

~~[illegible]~~ And by extending + and * to $\mathbb{Z}/m$ we get: (from the com. ring $\mathbb{Z}$)

$[a] + [b] = [a+b]$ & $[a][b] = [ab]$ ⇒ $[a]^{-1}$ exists iff $\exists [b] \neq [0]$, $[[a][b]] = [1]$

Thus $\mathbb{Z}/m$ is also a com. ring.

~~[illegible]~~ . $\mathbb{Z}/m$ respects the field axioms iff m is prime (or a prime power ~)

Hence $\mathbb{F}_m = \mathbb{Z}/m$ for all m prime $\mathbb{F}_{m^k}$

GARY: Possible to have inf. field w/ finite char. e.g. ra° polys with coefs from finite set.

Number system digression

$\mathbb{N} \xrightarrow{-} \mathbb{Z} \xrightarrow{/} \mathbb{Q} \xrightarrow{\surd} \overline{\mathbb{Q}} \xrightarrow{\lim} \mathbb{R} \xrightarrow{\cancel{\text{order}}} \mathbb{C} \xrightarrow{\cancel{* \text{commu}}} \mathbb{H} \xrightarrow{\cancel{* \text{assoc}}} \mathbb{O} \xrightarrow{\cancel{* \text{altern}}} \mathbb{S}$

𝕀 blackboard bold

. Vector space

<u>Def</u>. mathematical structure ~~that contains objects called vectors that are subjects to elements of a field through scalar multiplication &~~ adjunct to a field (~~module over a ring~~) affected by consisting of a set of elements called vectors, an addition opera° respecting commu, assoc, ident, invert operating on the vectors, and a multiplication operation defined over the set of vectors and the field called scalar multiplication resp. ident, dist over field +, dist over vector +, compatibility (≈assoc) w/ f of scalar * with field *.

Morphism digression

. homomorph - structure-preserving map
. isomorph - bijective homomorph
. homeomorph - topological isomorph

. automorph - self isomorph
. endomorph - self homomorph
. diffeomorph - isomorphism for smooth manifolds

Euclid's alg and its rela° to invertibility in Galois fields

GARY: By reversing Euclid's alg can find inverses!

Euclid e.g. gcd(a, b) ⇒ | gcd 16, 12
16 = 12 + 4
12, 4
12 = 4 × 3 + 0
→ ④, 0

→ Factor ~~(p+1)~~ ~~for~~ every ~~one of the~~ multiple of (p+1):

$p+1 = q \times R \Rightarrow [q]^{-1} = [R]$ (and use symmetry)

$2(p+1) = \ldots$

e.g. $\mathbb{F}_7$ $1 \times 1 = 1$ $(p-1)^2 = p^2 + 1$

7+1 = 2 × 4
14+1 = 3 × 5

LEC 02 - Linear systems and vectors : Matrix representa°

- Linear sys & matrices

Def: sys of equa° of linear polys with coefs in some field $\mathbb{F}$ -

$S = \begin{cases} a_{11}x_1 \dots = \alpha_1 \\ \vdots \end{cases} \to [S] = \left[\begin{array}{c|c} a_{11} \cdots & \alpha_1 \\ \vdots & \vdots \end{array}\right] = \begin{bmatrix} a_{11} \cdots \\ \vdots \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \end{bmatrix}$

The # rows is an upper bound on dim (solu° space)
Rows can be swaped without changing the ↑
Linear combos of rows can be taken ~~this~~, as long as nothing gets lost -

(~~→ Row ops preserve the nullspace~~) → Row ops preserve the sol° space (which is the nullspace when the sys is homogeneous)

→ Matrix equa°

$S = [Ax = b]$ · where A is coef mat
· x, b are col vectors

$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \boxed{x}\,v_1 + \boxed{y}\,v_2 + \boxed{z}\,v_3$

Getting Ahead

GARY: If we look at **A** as the function ~~from~~ $[x \mapsto b]$ then we realise that the sol° sp of every non homogeneous sys (i.e. $b \neq 0$) is a transla° of ~~that~~ the solu° of the homogeneous sys $Ax = 0$ (i.e. $b = 0$)

proof $Ax = 0$ for some x
$Ax' = b$ for some x'
By linearity : $A(x+x') = A(x) + A(x') = 0 + b = b$

Interpreta° : so if x' is a sol to non hom, then $\{x + x' \mid Ax$ is sol to hom$\}$ is the sol space to non hom -

→ Substitutions

[Omission: elementary row ops (premultiplica°) correspond to a special kind of substitu° → $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{matrix} R_1 \\ R_2 \end{matrix}$ $R_2' \leftarrow R_2 + R_1 \Leftrightarrow R_2' = (a+c)x + (b+d)y$]

Consider S_1, S_2 2 lin sys s.t. S_2 has as many equations as S_1 has variables, then substituting the vars in S_1 by their expressions in S_2 is isomorph to matrix multiplication -

~~e.g. $A_{m \times n}$, $B_{n \times k}$ $AB = C_{m \times k}$ where~~ $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$

e.g. $\begin{matrix} x + 2y + z, \\ x + y - z, \end{matrix}$ $\begin{matrix} x = e + f \\ y = e - f \\ z = e + 2f \end{matrix}$ → $\begin{matrix} \underline{(e+f) + 2e - 2f + e + 2f} = 4e + f \\ = e - 2f \end{matrix}$

- Linear sys & Vector spaces

The sol space to a lin sys can be parametrized in ≠ ways,

e.g. $x + y = 2$
$[1\ 1 \mid 2]$
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

A vector space is also parametrizable in ≠ ways (regardless of the basis it is the same space!)

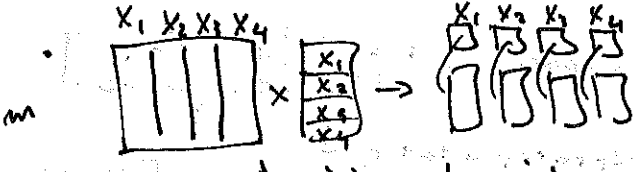
- The sol° space is either a point, line, plane etc -
  - sol° space to homogeneous has those going through the origin
  - → A subspace of a vector space has those same properties -
  - → i.e both have this closure property under linear combina° -
- Vector spaces will be a powerful tool to study linear ~~algebra~~ systems -
  - A vector space is only characterised by a dimension - ~~thensame~~ any set of coordinates / basis will produce isomorphic parametrizations
- Concretely: vector spaces (& subspaces) have the intrinsic properties of the sol spc to homogeneous lin sys -

Rowe: there is no 'ambient' space around it, its seen from the 'inside'

$\boxed{\text{Prop } (Ax = 0)}$		$\boxed{\text{Axioms of Vec spc}}$
$x_1, x_2 \in Sol$ $\Rightarrow x_1 + x_2 \in Sol$	⟷	Vector addition is binary (ie $V \times V \to V$) binary ops are closed. $\Rightarrow v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V$
There is always a trivial sol° (⇒ the 0 vector $\begin{bmatrix} 0 \\ \vdots \end{bmatrix}$)	⟷	$0 \in V$
$x_1 \in Sol$ $\Rightarrow \lambda x_1 \in Sol$ $(\lambda \in \mathbb{F})$	⟷	Scalar mult is binary (⇒ closed)

- more generally: ~~every~~ sol spaces and vector spc/subspc share the same closure props ~~for~~ under linear combinations -

- Vector spaces & matrices



cols lie in colspc of matrix

⟺ if [T] is LT, then cols lie in the range

$$T : \overset{\leftarrow \text{domain}}{V_m} \to \overset{\leftarrow \text{codomain}}{V_n}$$

V_m ⊎ Row space

V_n ⊎ colspace

codomain = range ⇒ T is 1-1

GARY: think of cols already in the range (⊆ codomain)

* bounded by height of cols (# rows) and number of cols

vectors in Vm

restricts basis

→ Computing the range is ~~finding~~ constructing a subspace in the codomain whose dimension is *

LEC03 - Change of bases & coordinates

- Possible subspaces of V_n

e.g. $V_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x,y \in \mathbb{F} \right\} = \left\{ x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \text{span}_{\mathbb{F}} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

induce subspaces through equa°

Matrix equa°:
- none $[0\ 0] \Rightarrow$ ~~the~~ the plane V_2
- 1 equ $\begin{cases} [1\ 0] \\ \text{OR} \\ [0\ 1] \end{cases} \Rightarrow$ a line in V_2 ⟵ ROWE they are the same ~~set~~ subspace (i.e. V_1) but ≠ subspaces of V_2.
- 2 equ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ a point in V_2

differ by RREF

- Two perspectives on subspace
  - As span of a basis
  - As space of sol to lin sys

e.g. • Let $W \subseteq V_3$ s.t. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} , \begin{matrix} x - y - z = 0 \\ 2y + z = 0 \end{matrix} \right\}$

Right now ~ is RREF
Later ~ is similarity

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{bmatrix} \Rightarrow W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} x - \frac{1}{2}z = 0 \\ y + \frac{1}{2}z = 0 \end{matrix} \right\}$

~~$\Rightarrow W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} 2x - z = 0 \\ 2y - z = 0 \end{matrix} \right\}$~~ [illegible] $= \left\{ z \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} : z \in \mathbb{F} \right\}$

Lin sys → vecspc basis

$W = \text{span} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

• Let $W \subseteq V_3$ s.t. $W =$ ~~$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right.$~~ $\text{span} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

We need a sys w/ 2 equa° s.t. 1 free variable (parameter of sol spc) ← e.g. z

$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} , \begin{matrix} x + az = 0 \\ y + bz = 0 \end{matrix} \right\}$, and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \in W$ thus: $\begin{cases} 1 + 2a = 0 \\ -1 + 2b = 0 \end{cases} \Rightarrow \begin{cases} a = -1/2 \\ b = 1/2 \end{cases}$

$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} x - \frac{1}{2}z = 0 \\ y + \frac{1}{2}z = 0 \end{matrix} \right\}$

vecspc basis → Lin sys

$W = \text{Sol} \left\{ \left[\begin{matrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \end{matrix} \right] \right\}$

change of $\boxed{\text{basis}}$ matrix

• Changing coordinates (will be revisited as it is a mess!)

ROWE

$$\begin{tikzcd} P^{\beta}_{\alpha} \arrow[r, leftrightarrow, "\text{INV}"] \arrow[d, leftrightarrow, "T"] & P^{\alpha}_{\beta} \arrow[d, leftrightarrow, "T"] \\ Q^{\beta}_{\alpha} \arrow[r, leftrightarrow, "\text{INV}"] & Q^{\alpha}_{\beta} \end{tikzcd}$$

change of $\boxed{\text{coord.}}$ mat.

Let $\alpha = \{v_1, v_2\}$ be a basis for V_2. An element of V_2 is written under that basis

$[v]_\alpha = \begin{bmatrix} a \\ b \end{bmatrix}$ ⟵ coordinates for the basis

Let $\beta = \{w_1, w_2\}$ s.t. ~~$\begin{cases} w_1 = 2v_1 + 0v_2 \\ w_2 = 1v_1 + 1v_2 \end{cases} \Rightarrow [v]_\alpha = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} [v]_\beta$~~ ← B in terms of α gives way to change B to get α

$\begin{cases} w_1 = \boxed{a_1 v_1 + a_2 v_2} \\ w_2 = \boxed{b_1 v_1 + b_2 v_2} \end{cases}$ ~~$\begin{cases} v_1 = \frac{1}{2} w_1 + 0 w_2 \\ v_2 = -\frac{1}{2} w_1 + 1 w_2 \end{cases} \Rightarrow [v]_\beta = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} [v]_\alpha$~~ Die

~~[illegible]~~ Let $[v]_\beta = \begin{bmatrix} x \\ y \end{bmatrix}$. Thus $v = x w_1 + y w_2$ so bad die ~~$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2$~~

$[v]_\alpha = \begin{bmatrix} x a_1 + y b_1 \\ x a_2 + y b_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ | $v = x a_1 v_1 + x a_2 v_2 + y b_1 v_1 + y b_2 v_2$

$v = \boxed{(x a_1 + y b_1)} v_1 + \boxed{(x a_2 + y b_2)} v_2$

LEC04 - The Fundamental Theorem of Linear Algebra

- Intuition
  - → A real vector space is a vector space over $\mathbb{R}$.
    - Consider V_3 over $\mathbb{R}$ (aka $\mathbb{R}^3$.) — does not need to have the 'Euclidian' orthonormal basis –
    - Consider the lin map T : [illegible] $\mathbb{R}^3 \to \mathbb{R}^3$
      - 4 cases for kernel shape:
        - 1) $T = I_n \Rightarrow$ all cols are lin. indep. $\Rightarrow$ Range = codomain
        - 4) $T = O_n \Rightarrow$ range $= 0$

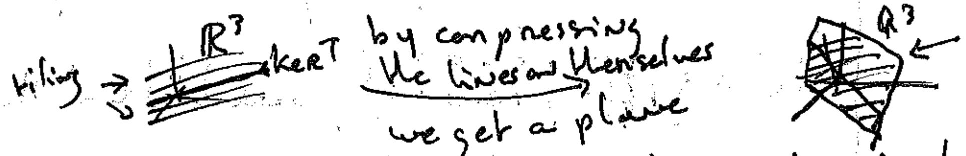
for 2) & 3) we have the fact that T brings at least 2 $^{v\neq}$ things in domain to 1 thing in range which is not 0.

Thus, this point in the range, call it b, is [illegible] the image of a shape in the domain which is the kernel translated –

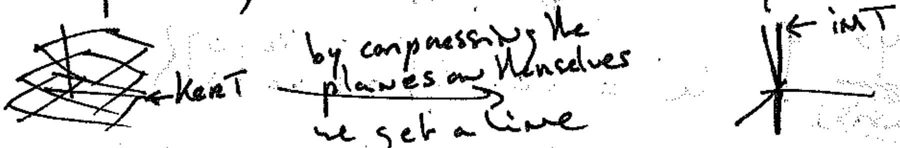
⚠ { Recall that everything in the domain that points to a single point in the range has the same shape as the kernel of T

$\boxed{\text{What goes to 0 under T} \Leftrightarrow \text{nullspc of } [T]}$

2) Hence if the kernel is a line, everything that points to a single point is a line → the domain can be tiled with line



3) if kernel is plane, we can fill the domain by stacking planes



* → i.e. the way the kernel is stacked is the range – ⚠

- → Another look at row ops: . they preserve the kernel (in domain)
  . they perform a LT on the range:

$B(Ax)$ → Ax → A pt x in the range; B → B changes shape of ~~[illegible]~~ the range of A

And this only works for homogeneous sys bc 0 is the fixed point

→ All other sols to non hom can be derived from hom

→ Direct sum notation:

- V is the direct sum of subspaces $(W_i)_m^i$ iff $W_1 \cap W_2 \dots W_m = \{0\} \wedge$ $\forall v \in V$, v can be written as a linear comb of vecs in $(W_i)_m$.
- Equivalently: $V = W_1 \oplus W_2 \dots W_m \Leftrightarrow$ Bases for $(W_i)_m$ can be combined to form a basis for V.
- Preserves vecspace axioms. e.g. $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$
  $(a,b) \leftarrow a, b$ such that: $\forall (a,b),(c,d) \in \mathbb{R}^2, (a,b)+(c,d) = (a+c, b+d)$

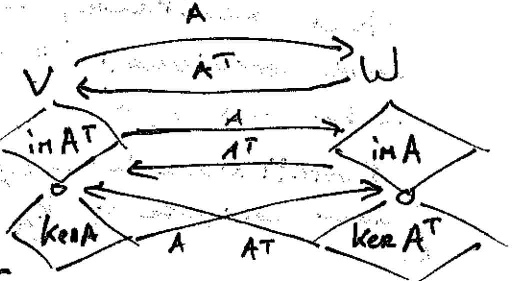
($\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ in this case cartesian prod & direct sum are same)

→ Dimension theorem

$T: \overset{n}{V} \to \overset{m}{W}$, $A = [T]$

- $\underbrace{\text{Rank } A}_{}$ = dim Im T ; $\overset{|null(A)|}{\text{nullity } A}$ = dim Ker T
  Rank A + nullity A = dim V ← (n, # of cols)
  (# of lin. indep. cols. = # lin indep. rows)

Recall from LinAlg 1:
- Row spc A ⊕ null A = V
  ⇒ rank A + nullity A = n
- ~~row~~ col spc A ⊕ null A^T = W
  ⇒ Rank A + nullity A^T = m



Asif: finding bases

$A = \begin{bmatrix} — R_1 — \\ — R_2 — \\ — R_3 — \end{bmatrix}$

REF ↓

basis (rowspc) $\left\{ \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{} \right.$

thus also basis

$A = \begin{bmatrix} | & | & | & | \\ \underbrace{v_1 \quad v_2 \quad v_3}_{*basis*} & & & v_4 \\ | & | & | & | \end{bmatrix} \xrightarrow[\text{(preserves col space)}]{\text{CEF}}$ would give basis straightaway and range!

& equivalently: REF(A^T) = CEF(A).

← REF (preserves row space) and kernel!

Since row rank = col rank, we know those first 3 cols came from lin. indep. cols. But since REF 'scrambles' the range (performs a LT) we can't use them as a basis for the original matrix, but we can identify which cols were originally responsible for lin. indep. thus *basis* is a basis for colspc

⇒ REF → $\boxed{\text{~~pre mult~~}}$ → LT on range, CEF → $\boxed{\text{~~post mult~~}}$ → LT on range of transpose

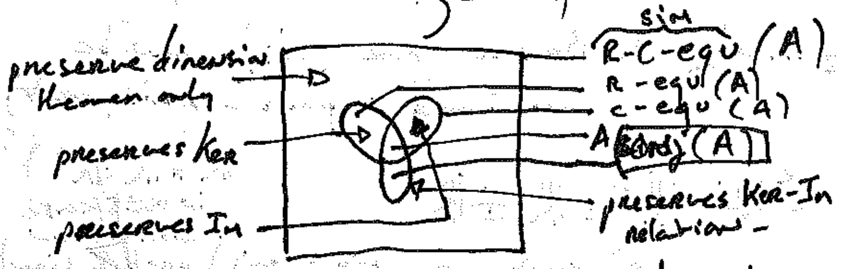
Rowe: all that's left is the dimension theorem

GARY+ROWE: pre/post are confusing! talk about left mult or right mult.

LECOS – Similarity

- We saw that we only need to study homogeneous systems as we need only to translate (no pun) the sol° spc (aka null spc)
- In the matrix representation of the system, we noticed that the linear system (rows) can be viewed as a vector space, but more importantly that the col vecs lie in another vec spc, that 'relies' on the first
- And thus a matrix, viewed as a LT, is a link between the domain (rows) and the codomain (cols)

⚠ Conjugacy is not R-, C-, RC equiv. It's equiv. to chg of coord.
ROWE: Similarity for LTs is RC-equiv.
- Change of coordinates is a bijection



Another note from ROWE: Rational Canonical Form theory & spectral theorem are Ⓐ but capture behaviours w/ ≠ relevance.

'the most fundamental results'

Note from ROWE: in undergrad I also conflated LTs and LOs thus viewing conjugacy (for LO) as a special case of similarity for square matrices.
But I realised there is a clear distinction between ~~both, and they should be viewed as~~ their behaviours ≠ and thus should be distinguished accordingly.

→ Similarity.
Given arbitrary matrix A m×n, and underlying (left mult)* LT $T: V \to W$, ~~[illegible]~~ ~~[illegible]~~ such that $[T]^\beta_\alpha = A$ for some bases α ~~[illegible]~~ and β resp. for V, W.

[*GARY: notation is L_A, do not confuse w/ R_A]

↳ this matrix fully describes T. Its columns are the $[T(\alpha)]_\beta$ that is the images of the basis vectors in terms of β. $[T]^\beta_\alpha = \left[\begin{bmatrix} | \\ T(\alpha_1) \\ | \end{bmatrix}_\beta \begin{bmatrix} | \\ | \end{bmatrix} \cdots \right]$
(coordinate-matrix of T ~~[illegible]~~ w/respect to α and β.)

GARY: in ~~the~~ platonic sense, $[T]_\alpha$ is enough to fully describe T. That is, how it changes a single fixed basis of V. But $[T]^\beta_\alpha$ is necessary to instantiate it (represent it) as a matrix.
(Alex). and T on its own is even more platonic. Just as a basisless vector space, it exists in the platonic world of Abstraction. Mathematics is about formalising these rational concepts, Metamathematics (<3) is understanding how this formalisation works

So keep in mind that $\text{Ker } T = \{v \in V \mid T(v) = 0\}$ ~~[illegible]~~
and as soon as we instantiate it for some bases α, β, we get:

$$\text{Ker } T = \text{null } [T]^\beta_\alpha$$

Thus, $\forall v \in V$, $\underbrace{[T(v)]}_{m \times 1}{}_\beta = \underbrace{[T]^\beta_\alpha}_{m \times n} \underbrace{[v]}_{n \times 1}{}_\alpha$

- If ~~[illegible]~~ ~~about~~ the solutions ~~to~~ to the homogeneous underlying linear system, Then Ker T can be represented as any element in the class of matrices R-eq to $[T]^\beta_\alpha$, i.e.:

$$\text{Ker } T = \text{null } [T]^\beta_\alpha = \text{null } [T]^{\beta'}_\alpha = \ldots$$

. Call this class $REQ_{[T]_\alpha^\beta} = \{M \mid M \text{ is R-eq to } [T]_\alpha^\beta\}$

. Then any element of $REQ_{[T]_\alpha^\beta}$ can be written as : $M = Q[T]_\alpha^\beta$ where Q is invertible.

also & ness see page $\boxed{10}$

. Geometrically, this corresponds to ~~finding~~ changing the way we 'measure' in the codomain, i.e. its basis. Thus if we ~~take~~ express a new basis β' in terms of β (i.e. $\beta'_1 = a_{11}\beta_1 + a_{12}\beta_2 \dots$, $\beta'_2 = a_{21}\beta_1 + a_{22}\beta_2 \dots$), and take the corresponding change of bases matrix $P_{\beta'}^\beta = [a_{ij}]$, we can obtain the change of coordinate matrix $Q_{\beta'}^\beta = (P_{\beta'}^\beta)^T$ (see "proof" on page $\boxed{5}$)

. Putting it all together we get $[T]_\alpha^{\beta'} = Q_\beta^{\beta'}[T]_\alpha^\beta$

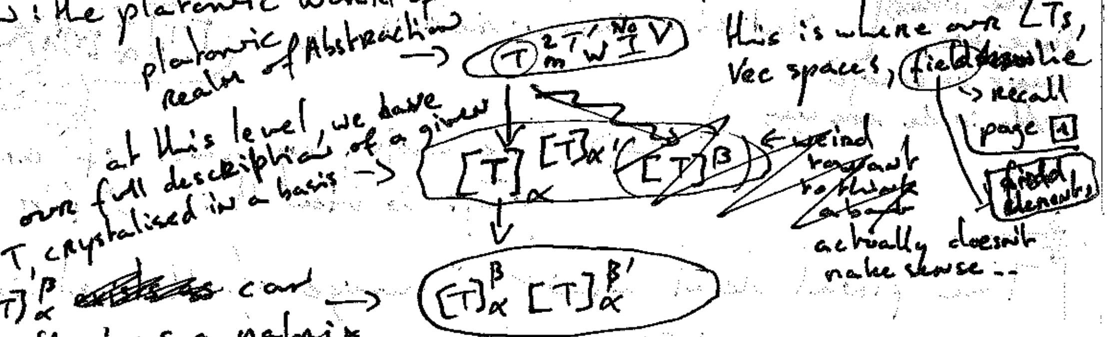
. Similarly, if we care about ~~the range~~ preserving our now platonic 'view' of the range (that is our basis), we get the countable set $\{M \mid M \text{ c-eq } [T]_\alpha^\beta\}$ and for any new basis α' for our domain, the rela° $[T]_{\alpha'}^\beta = [T]_\alpha^\beta Q_{\alpha'}^\alpha$ (notice in this case, as it is right multiplication, ~~that~~ we need the inverse of $Q_\alpha^{\alpha'} = (P_\alpha^{\alpha'})^T$, i.e. $Q_{\alpha'}^\alpha = (Q_\alpha^{\alpha'})^{-1}$)

. Another view on theses 'change of bases matrices' (or their transposes for coordinates) is presented here:

- we mentioned that we can view left multiplication as performing an LT on the range of an LT. (same for Right Mult. and kernel)
- we want our 'change of coordinate matrices' to be invertible
- we can view them as instantiations of the identity transform! That is, we can view our equivalence class (same for $CEQ_{[T]_\alpha^\beta}$) ~~as composed~~ as the set $REQ_{[T]_\alpha^\beta} = \{Q[T]_\alpha^\beta \mid Q = [I]_\beta^{\beta'}$ for some β', a basis for W$\}$

Long awaited digression: the platonic world of Abstraction for LTs.

Meta digression:
in math, we grab stuff from up there and use it.
in metamath, we study ~~the~~ ways to grab and implement those things



Finally, at this level $[T]_\alpha^\beta$ ~~exists as~~ can be represented 'physically' instantiated, as a matrix.

- in other words, when we left multiply by an invertible matrix (sim, for Right), we perform an isomorphism on the range.

. Def 2 matrices A, B of the same size are similar iff $\exists P, Q$ invertible, such that $A = PBQ$.

Property Any matrix in an equivalence class defined by similarity shares the same RRCEF.

proof: dimension theorem

$$A \in RREQ_B \Rightarrow RRCEF(A) = RRCEF(B) = \begin{array}{c|c|c} & k & m-k \\ \hline k & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline m-k & 0 & 0 \end{array}$$

k is the rank

m-k is the nullity

→ ROWE: all that remains is the dimension formula _

→ Conjugacy

. Can be seen as a special case of similarity, but better seen as the counterpart of similarity for LOs.

$LT : V \to W$ (linear) mapping between vec spaces

$LO : V \circlearrowleft$ endomorphism (twisting a vec space)

Def Two matrices A, B are conjugate iff $A = Q^{-1}BQ$

→ Hence $\boxed{\text{iff } L_A = L_B}$ (the platonic underlying LOs are the same)

proof: ~~Let~~ [illegible]. Let α, β be bases for V.

$\Rightarrow$ $L_A(v) = [L_A]_\alpha^\alpha [v]_\alpha = A[v]_\alpha$ (LO are basisless)

$= QB(Q^{-1}[v]_\alpha) = QB[v]_\beta$ $(Q^{-1} = Q_\alpha^\beta)$

$= Q[L_B(v)]_\beta = [L_B(v)]_\alpha = L_B(v)$ $(Q = Q_\beta^\alpha)$

this was not a proof as it is using what we are trying to prove, but gives intuition maybe

(Alex) think of QAQ^{-1} as Q changing the basis of the domain of L_A while Q^{-1} is acting on the basis of the domain of L_A, and they are inverses since we want to change both in the same way

Platonic world of LOs → $T_1 \; T_2 \; T_3$

Matrix instantiations → $[T_1]_\alpha^\alpha \; [T_1]_\beta^\beta$ ← conjugacy class of T_1

⇒ The conjugacy class of an LO captures all possible ways to view this LO (in a single basis).

The final deal about "change of bla matrix"

express α' in terms of α

$$\alpha' \downarrow [a_{ij}] \xrightarrow{\alpha} \Rightarrow \nabla_{\alpha'}^{\alpha}$$

knowledge of α gives us

(chg basis)

$T \downarrow$

(chg coord)

$$\left(\nabla_{\alpha'}^{\alpha}\right)^T = \Delta_{\alpha'}^{\alpha}$$

← we can get smthg in α

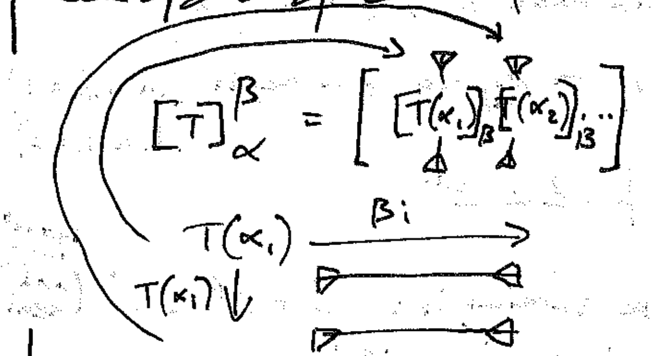
← If we know smthg in α'

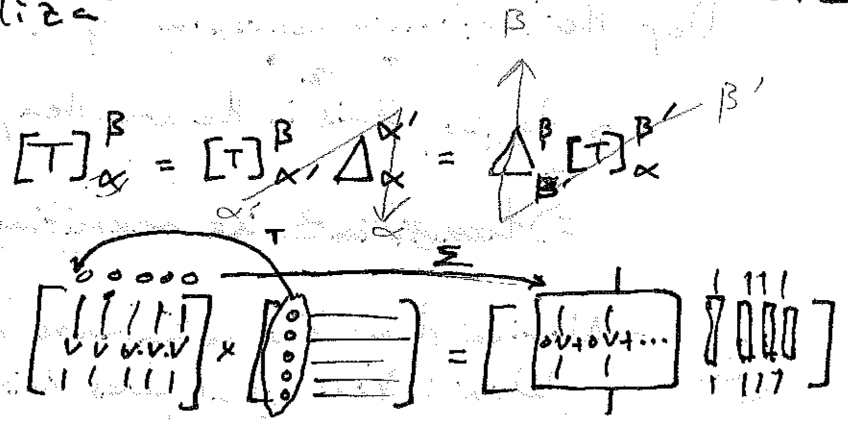
→ i.e. $[v]_\alpha = \Delta_{\alpha'}^{\alpha} [v]_{\alpha'}$

$$\begin{tikzcd} \text{Bases:} & \nabla_{\alpha'}^{\alpha} \arrow[r, leftrightarrow, "-1"] \arrow[d, leftrightarrow, "T"'] & \nabla_{\alpha}^{\alpha'} \arrow[d, leftrightarrow, "T"] \\ \text{Coord:} & \Delta_{\alpha'}^{\alpha} \arrow[r, leftrightarrow, "-1"] & \Delta_{\alpha}^{\alpha'} \end{tikzcd}$$

$$[T]_\alpha^\beta = \cancel{\Delta_{\alpha'}^{\alpha}} [T]_{\alpha'}^{\beta} \Delta_{\alpha}^{\alpha'}$$

~~Graphically this~~ Useful visualiza:





Sorry, but another digression ....

CALC I

- contravariance e.g. $f(x+a)$ → adding a ⇒ ⊖ shift
- covariance e.g. $x \mapsto f(x)+a$ → adding a ⇒ ⊕ shift

LINALG

coords – basis

(double basis vec, diff resp. coor. by 2 –

LEC06 - The Spectral Theorem

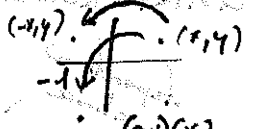
→ The complex field

. Let $\mathbb{C} = V_2$ over $\mathbb{R}$, with complex mult. defined as bla –

<u>Defs</u> . The space of 2x2 matrices over $\mathbb{R}$: $\{M_{2\times2} \mid M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \text{ for } a, b \in \mathbb{R}\}$

↳ corresponds to rotation and scaling (vectors are not 'rooted')

e.g. $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ → (i.e. $(a+ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$)



$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -x+y$

Notice that $A + A^T = D$ ← diagonal (i.e. scales well)

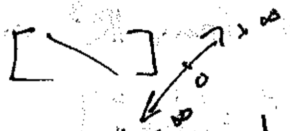
① ↳ this behaviour of 2x2 matrices encapsulated in $A + A^T = D$, leads to the fundamental theorem of Algebra

(i.e. $\mathbb{C}$ is algebraically closed)

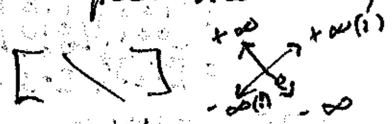
→ The conjugate transpose

. Consider symmetry for real matrices i.e. A sym. iff $A = AT$

→ symmetric matrices over $\mathbb{R}$ have 1 axe of symmetry ($\mathbb{R}^1$)



. Extend the concept to complex matrices, they will have 2 axes of syn ($\mathbb{R}^2$)



good mathematical argument? <u>NO</u>

→ intuitively, it makes <u>perfect sense</u> to generalise sym. to get <u>hermity</u> (sym. becomes sp. case for reals)

Def the conjugate transpose of a matrix Z over $\mathbb{C}$ is $Z^* = (\bar{Z})^T$
2

→ Again, this is the counterpart of transpose for reals ($\dim \mathbb{R} = 1$)
1 ($\dim \mathbb{C} = 2$)

⇒ A matrix Z is hermitian iff $Z = Z^*$

② → Notice that the behaviour of those matrices is: $Z + Z^* = S$ ← symmetric (and real)
Wow! Now what does this behaviour give us?
(→ the spectral theorem)

① + ② → ~~[illegible]~~ proof by intuition: apply ① (the fund. theorem of Algebra) to characteristic poly of hermitian matrices ②.

⇒ $\boxed{\text{The Spectral theorem}}$

M is hermitian iff all of M's eigvals are real ∧ M is diagonalizable

Def
- M is positive definite iff all of M's eigvals are in $\mathbb{R}^{*+}$ ∧ M is diagonalizable
- M is positive semidefinite iff ——— $\mathbb{R}^+$ ∧ ———

Digression: where do the rows of a matrix live?
(Alex) They must be in some space kinda constraining the vecs in V?
GARY + ROWE: They live in the dual space:

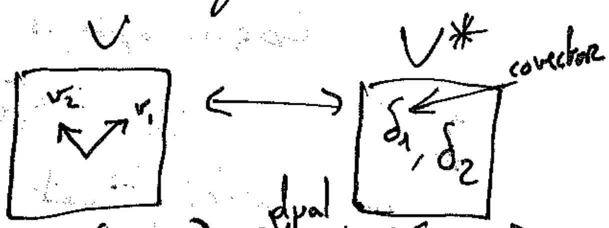
$V^* \overset{\text{def}}{=} \{\text{LTs from } V \text{ to } \mathbb{F}\}$

depends on a choice → $V \cong V^{**}$ ← but definitely isomorphic

there is a bijec° from V to V^*

$$B : V \to V^*$$
$$v \mapsto f_v : V \to \mathbb{F}$$

(Alex) And it goes without saying that $f_v(\text{span } v) = \mathbb{F}$.



basis: $\{v_1, v_2\}$ ⇒ dual basis $\{\delta_1, \delta_2\}$

where: $\delta_1 : V \to \mathbb{F}$, $\delta_1(v_1) = 1$, $\delta_1(v_2) = 0$ | $\delta_2 : V \to \mathbb{F}$, $\delta_2(v_1) = 0$, $\delta_2(v_2) = 1$ ← amount of v_2 in direc° of v_2

⇒ $\dim V = \dim V^*$

GARY's example for appreciating platonic LTs

. Consider T_1, T_2 LTs from $\mathbb{R}^3$ to $\mathbb{R}^2$ w/ bases α for $\mathbb{R}^3$, β for $\mathbb{R}^2$

$T_1(\alpha_1) = \beta_0$
$T_1(\alpha_2) = \beta_0$
$T_1(\alpha_3) = \beta_1$

$$\Rightarrow [T_1]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [T_2]_\alpha^\beta \, [\cdot] \leftarrow \text{col op}$$

⇓ change basis of T_2's domain

$T_2(\alpha_1) = \beta_0$
$T_2(\alpha_2) = 0$
$T_2(\alpha_3) = \beta_1$

$$\Rightarrow [T_2]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

} T_1 projects span$\{\alpha_1, \alpha_2\}$ on 1 dimension (span β_0), whereas T_2 collapse an entire span α_2 to the 0 vector.
However T_1 & T_2 are the same platonically, this can be seen by changing the α_2 direc° to $\alpha_1 + \alpha_2$ (i.e. the (invertible) col. op.)

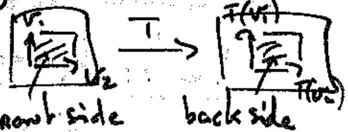
LEC07 - Generalised Eigenspaces

But they behave differently LTs up to similarity → LOs up to conjugacy

→ The determinant

(As seen in the previous example LTs exist regardless of their bases) - For LOs that are studied with a single basis (domain = codomain), we can measure how much the LO distorts "the oriented area/$\boxed{\text{volume}}$ of a basis tile"

e.g. $T(v_1) = v_2$, $T(v_2) = v_1$



. Det captures smthg platonic, it is invariant given a conjuacy class ~~[illegible]~~.

i.e. $\det(Q^{-1}AQ) = (\det Q^{-1})(\det A)(\det Q) = \det(A)\det(Q^{-1}Q) = \det(A)$

→ The trace

. Defined for sqr matrices (underlying LO) just as det; also conjugacy class invariant (thus also captures smthg 'platonic' about LOs)

Def: $tr(A) = \sum_i^n A_{i,i}$ (i.e. sum of elements of main diagonal)

$\Rightarrow$ tr is a linear map : $\begin{cases} tr(A+B) = tr(A) + tr(B) \\ tr(cA) = c\,tr(A) \end{cases}$

[Note: A & A^T have the same diagonal elements: $tr(A) = tr(A^T)$

Thm: $tr(AB) = tr(BA)$

thus : $tr(Q^{-1}AQ) = tr(A)$

$x^3 + \square x^2$

tr → -free space!-

→ $x^4 + \square x^8$

[Note: unlike det, $tr(ABC) \neq tr(ACB)$ in gen. But tr is invariant for cyclic permuta°, e.g. $tr(ABC) = tr(CBA)$

→ The Characteristic Polynomial

. **Def**: $c_T(x) = \det(T - xI)$ $\Rightarrow$ polynomial over $\mathbb{F}$, with leading coef $(-1)^n$

ROWE : it's a poly, you can plug in matrices or LOs up (i.e. $c_T(T) \neq \det(T - TI)$. if ~~[illegible]~~ $c_T(x) = p(x)$ then $c_T(T) = p(T)$)

Thm: $\forall T :: LO$, $c_T(T) = 0_V$ (sim for matrices), i.e. T is annihilated by it char poly.

. **Def**: $m_T(x)$ is the smallest degree poly ~~over~~ $\mathbb{F}$ that has leading coef 1 and annihilates T (aka monic)

$\Rightarrow m_T(x)$ has same roots as $C_T(x)$

$\Rightarrow m_T(x)$ divides $C_T(x)$

Thm: If $C_T(x)$ has a root λ in $\mathbb{F}$, $\exists$ λ-eigvec of T -

proof: ($\lambda \in \mathbb{F}$) $c_T(\lambda) = 0 \Leftrightarrow \det(T - \lambda I) = 0 \Leftrightarrow T - \lambda I$ is not invertible (i.e. not an isomorphism)

$\Leftrightarrow \dim Ker(T - \lambda I) \geq 1$ (not an isomorphism $\Rightarrow \exists$ vecspc that goes to 0)

$\Leftrightarrow \exists\, v$ nonzero s.t. $(T - \lambda I)(v) = 0_V$ (i.e. v is a λ-eigvec)

. **Def**: the previous proof showed us where the λ-eigvecs live, i.e.

$E_\lambda = Ker(T - \lambda I)$ is the λ-eigenspace of T iff $E_\lambda \neq 0_V$ (iff λ is a root)

→ Irreducibility

. **Def**: a polynomial is irreducible over a field $\mathbb{F}$ if it cannot be factored.

$\Rightarrow$ Over $\mathbb{C}$, only linear polys are irreducible (Fund. Theorem of Algebra)

$\Rightarrow$ Over $\mathbb{R}$, linear polys and quadratic polys w/ no real root are irreducible

→ Multiplicities of eigenvalues

. algebraic : exponent of eigval λ in corresponding factor of char poly, i.e. $(x-\lambda)^{\overset{\frown}{a_\lambda}}$

. geometric : $\dim E_\lambda$

—▷ Generalised Eigspc

<u>Def</u>: Given λ, an eigval of T, the generalised λ eigspc is $K_\lambda = \{v \in V \mid (T-\lambda I)^k v = 0\}$ → for some $k \in \mathbb{N}$

<u>Thm</u>: For a given λ, eigval of T, $\dim K_\lambda = a_\lambda$ (algebraic multiplicity)

⇒ for an eigval λ : $\text{geom}(\lambda) \geq \text{algb}(\lambda) \geq 1$ ⇒ $\mathcal{E}_\lambda \overset{\text{subspace Nota:}}{<} K_\lambda$

—▷ Cyclic subspaces

<u>Def</u>: $K \leq V$ is T-invariant iff $T(K) < K$

<u>Def</u>: $K < V$ is T-cyclic iff K is T-invariant & $\exists k \in \mathbb{N}, \exists v \in V, K = \text{span}_{\mathbb{F}} \overbrace{\{v, \dots T^k v\}}^{\text{cycle}}$

<u>Thm</u>: For λ, eigval of T, K_λ is T-invariant (also $(T-\lambda I)$-invariant by definition)

<u>Def</u>: The <u>cyclic subspace</u> generated by v (for LO T) is $C(v) = \text{span}_{\mathbb{F}}\{v \dots T^k v\}$ such that $\{v \dots T^k v\}$ is lin indep and $\{v, \dots T^k v\} \cup \{T^{k+1} v\}$ are not

⇒ it is a T-invariant subspace

—▷ Annihilator subspaces

<u>Def</u>: $K_{p(x)} = \{v \in V \mid p(T) v = 0\}$ ← the most general we can get: $\mathcal{E}_\lambda \to K_\lambda \to K_{p(\lambda)}$

<u>Thm</u>: $C(v)$ is an annihilator subspace of V for $p(x)$.

<u>proof</u>: by def, $T^{k+1}(v) = \sum_{i=0}^{k} (a_i T^i v)$

$\Rightarrow T^{k+1}(v) - \sum a_i T^i(v) = 0 \Rightarrow (T^{k+1} - \sum a_i T^i)(v) = 0$

Let $p(x)$ = substitute x for T in $\wedge$ ($\Leftrightarrow$ substitute T by x)

Then $C(v) = K_{p(x)}$

LEC08 - Rational Canonical Form Theory

—▷ **Diagonalisation**

T is diagonalizable iff $c_T(x)$ <u>factors completely</u> over $\mathbb{F}$ and for every eigenval λ, $\text{algb}(\lambda) = \text{geom}(\lambda)$

⇒ if $c_T(x)$ factors into <u>distinct</u> <u>linear</u> factors over $\mathbb{F}$, then T is diagonalizable

T is diagonalizable iff $\forall \lambda$ :: eigval of T, $\mathcal{E}_\lambda = K_\lambda$

—▷ Jordan Canonical Form & Rational Canonical form

• <u>Def</u>: N :: LO is <u>nilpotent</u> iff $\exists k \in \mathbb{N}^*, N^k = 0$

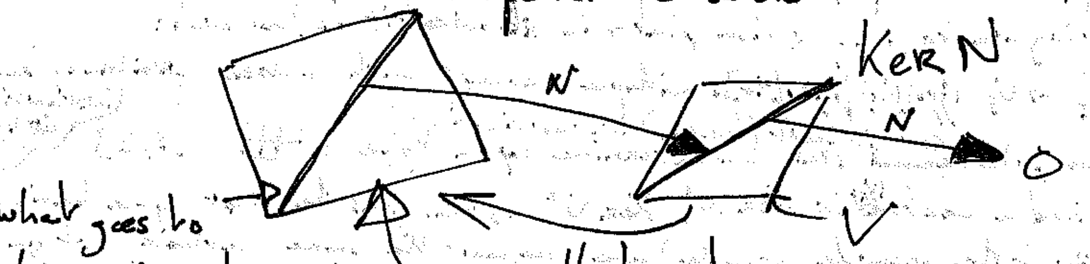
Thus N is nilpotent iff $m_N(x) = x^k$ for some k

iff $c_N(x) = (-1)^n x^n = (0-x)^n$

iff 0 is N's only eigval

iff $\forall k \in \mathbb{N}^*, \text{tr}(N^k) = 0$

How nilpotents work: (read from right to left)



$N : V \to V$

pull back

$(V/\ker N)$ ◁ Modding out:

$(V_1 < V_2)$, $V_1 \oplus V_2/V_1 \cong V_2$

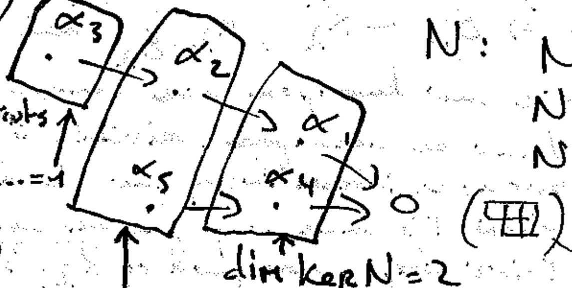
$V_2 \bmod V_1$

$\boxed{N \text{ is 1-1 on } V/\ker N}$ ◁ "although gross abuse of terminology"

ROWE: depends on a choice
GREY: not 'canonically' isomorph

+ So a nilpotent 'decomposes' its domain in such a way..

e.g. $V = V_5$, with basis $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$



This was a simple example w/ nilpotents, in general, to get 1s on the superdiagonal, need to find the image of the 'pulled back' basis → cycle basis!

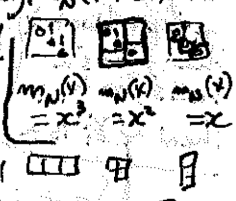
N: $N\alpha_3 = \alpha_2$ $N\alpha_5 = \alpha_4$
$N\alpha_2 = \alpha_1$ $N\alpha_4 = 0$
$N\alpha_1 = 0$

$\Downarrow$ $N(\alpha_1)$ $N(\alpha_2)$.....

$$[N]^{\alpha}_{\alpha} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

"can interpret this row as nothing goes to α_5"

+ There are thus a 'small' number of platonic nilpotents, operating on a finite vecspc V_m, this number captures how much applications of N it takes to annihilate the vecspc (this is the exponent of the minimal polynomial) and how big are its 'pulled back' kernels. This number is hence simply the # of partitions of m → e.g. $c_N(x) = x^3$



As mentioned informally, to get the canonical form of a nilpotent, we want a 'cycle basis': from previous example:

ROWE: define 'kernel stat' as follows: $(\dim \ker N^m, \dim \ker N^{m-1} \dots \dim \ker N^0)$

$$\begin{array}{|c|c|c|} \hline \alpha_3 & \alpha_2 & \alpha_1 \\ \hline & \alpha_5 & \alpha_4 \\ \cline{2-3} \end{array} \xrightarrow{N}\ 0$$

the span of these are cyclic subspaces, i.e. $K_0 = \mathcal{E}_0$

in general (Jordan) they are harder to find

"the jumps in kerstat can be used to determine the operator's cycle type (i.e. which partition of m it corresponds to)".

1, 2, 2 → 1 2 2

kerstat: (5, 5, 5, 4, 2, 0)

⇒ cycle type: [3, 2] (or [2, 3])

Note: the greatest # in the cycle type corresp. to the exponent of the minpoly, it is the size of the biggest cycle basis.

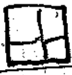
So nilpotents are a great way to partition a vecspc, and using cycle bases gives a nice canonical repr. of it.

→ Given a nilpotent $[N]_\gamma^\gamma$, we can determine to which canonical form it is conjugate (i.e. which platonic nilpotent it is) as such:

→ $N: V_5 \to V_5$ (just like previous example, except with a ~~more~~ arbitrary (instantiated w/ basis γ))

- compute kernel stat → determine cycle type :  [3,2]
- find a vector v_1 in $\text{Ker } N^3 / \text{Ker } N^2$ → this gives the first cycle basis $C(v_1)$
- to get the second (2-cycle) one, repeat the same method, but make sure $C(v_1) \cup C(v_2)$ is independent.
- the canonical basis for N of V_5 is thus $\beta = C(v_1) \cup C(v_2)$

$$\Rightarrow [N]_\gamma^\gamma = \Delta_\beta^\gamma [N]_\beta^\beta \Delta_\gamma^\beta, \text{ where } [N]_\beta^\beta = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & & 0 \\ & 0 & 1 & & 0 \\ \hline & & 0 & 0 & 1 \\ 0 & & & & 0 \end{array}\right]$$

free spc ⇒ note: we said 'the (Jordan) canonical form' but there can be several bc [3,2]~[2,3], i.e.  &  are canonical

Def: The Jordan canonical form (of a LO T) is a special matrix representation of T that is possible when $C_T(x)$ factors completely over $\mathbb{F}$ –

- algebraic mult need not equal geometric mult.

  $\Rightarrow \mathcal{E}_\lambda < K_\lambda$, $K_\lambda / \mathcal{E}_\lambda$ need not equal 0.

- We can decompose a vecspc into T-invariant subspaces. Every V can be decomposed trivially into improper T-inv subsp: V & O, but more interestingly can be decomposed into proper T-inv subspc. [e.g. when T's char poly factors completely over $\mathbb{F}$ (in which case, those T-inv subspc correspond to K_{λ_i}'s)]
- Notice that we can define nilpotents for each generalised eigenspc as such $N_{\lambda_i} = (T - \lambda_i I)$. This will allow us to construct cycle bases as previously shown.

  ⇒ those bases β_i can be unioned to get a Jordan basis β of T –

- This nilpotent analysis of T's generalised eigspc will construct the Jordan blocks of the Jordan CF; however this time each Jordan block will have a distinct λ_i on the diagonal (instead of 0s which might not be an eigval of T). The Jordan block of λ_i is formed as:

'T restricted to K_{λ_i}' (± notation $T|K_{\lambda_i}$ pipe)

$$[T_{K_{\lambda_i}}]_{\beta_i}^{\beta_i} = [N_{\lambda_i}]_{\beta_i}^{\beta_i} + [\lambda_i I]_{\beta_i}^{\beta_i} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & 0 \\ & & \lambda_i & \\ 0 & & & \lambda_i \; 1 \\ & & & \lambda_i \end{bmatrix} \leftarrow \text{from previous example}$$

$$\Rightarrow [T]_\beta^\beta = \bigoplus_{i=\#\text{eigvals}} [T_{K_{\lambda_i}}]_{\beta_i}^{\beta_i}$$ → with $\beta = \bigcup_i \beta_i$ & extension of ⊕ to matrices –

We are jumping steps but hopefully there is enough material to build a good intuition

Thm:
Let T be LO s.t. $C_T(x)$ factors completely.
Then:
- $V = \bigoplus_i K_{\lambda_i}$
- K_{λ_i} is T-inv.
- For each i, let γ_i be a union of cycles of generalised eigenvectors corresponding to λ_i that forms a basis for K_{λ_i}.

Then $\bigcup_i \gamma_i$ is a Jordan Canonical basis for T.

Def (+Thm): The Rational Canonical Form (a.k.a. Frobenius Normal form ~) is a special matrix rep. of LO T, that is unique (up to permut.). Thus every platonic LO has a unique RCF.

Diag: $V = \bigoplus_i \mathcal{E}_{\lambda_i}$
JCF: $V = \bigoplus_i K_{\lambda_i}$
RCF: $V = \bigoplus_i K_{p(x)}$

In RCF, we want to decompose those annihilator subspaces until they become T-cyclic

Def: suppose $K_{p(x)}$ is T-cyclic, the companion matrix $A_{p(x)}$ of $p(x) = x^d + \alpha_{d-1}x^{d-1} + \dots \alpha_1 x + \alpha_0$

$$A_{p(x)} \overset{\text{def.}}{=} \begin{bmatrix} 0 & 0 & \cdots & -\alpha_0 \\ 1 & 0 & \cdots & -\alpha_1 \\ 0 & 1 & \cdots 0 & \vdots \\ 0 & \cdots & 0 \; 1 & -\alpha_{d-1} \end{bmatrix}$$

Let $V = K_{p_1(x)} \oplus \dots$ (T-cyclic) then ∃ a basis β such that:

$$[T]_\beta^\beta = \begin{bmatrix} \boxed{A_{p_1}} & \boxed{B} & & 0 \\ 0 & \boxed{A_{p_1}} & & \\ & & \boxed{A_{p_2}} \cdots & \\ 0 & & & \boxed{A_{p_k}} \end{bmatrix}$$ ← e.g.

where $\beta =$ [illegible]

⇒ Rational Canonical Form Theorem → Two (SQR) matrices A, B are similar (conjugate) iff RCF(A) = RCF(B)