LECO1 - Linear Systems, Matrix Algebra

- System of linear equations
  \[ S = \{ \sum_{i=1}^{\infty} a_i x_i = \alpha \} \]
  \[ S \text{ is consistent iff } S \text{ admits at least one solution} \]
  \[ S \text{ is inconsistent iff } S \text{ admits no solution} \]
  \[ \alpha, \beta, \ldots = 0 \Rightarrow S \text{ is homogeneous} \]

- Systems are equivalent if they have the same solution set.

- Matrix representation:
  \[ \begin{bmatrix} a_{m1} & \cdots & a_{mn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \]
  \[ [A] \]
  \[ \begin{bmatrix} a_{m1} & \cdots & a_{mn} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \alpha \\ \vdots \\ \alpha \end{bmatrix} \]
  \[ [A] \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \alpha \\ \vdots \\ \alpha \end{bmatrix} \]

- Linear combination of column vectors:
  \[ x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ \vdots \\ a_{2n} \end{bmatrix} + \cdots = \begin{bmatrix} \alpha \\ \vdots \\ \alpha \end{bmatrix} \]

- Matrix algebra
  - Addition: commutative, associative, neutral element: zero matrix \([0] \)
  - Scalar product: commutative, associative, distributive \((\cdot)\)
  - Transposition:
    - \( A^T = [a_{ij}] \) is the transpose of \( A = [a_{ij}] \) iff \( [A]_{mj} = [A^T]_{ij} \)
    - \( (A^T)^T = A \), \( (A + B)^T = A^T + B^T \), \( cA^T = (cA)^T \), \( (AB)^T = B^T A^T \)
  - Multiplication:
    - \( AB \) is the product of \( A_{m \times n} \) and \( B_{n \times p} \) iff \( [AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \)
    - \( AB \) is non-commutative, neutral element: identity matrix \( I_n \)
    - Resulting matrix holds dot product of rows of \( A \) and columns of \( B \)
  - Exponentiation:
    - \( A^0 = I_n \), \( A^p A^q = A^{p+q} \), \( (A^p)^T = A^p \), \( (AB)^p = A^p B^p \) \( \forall p \in \mathbb{N} \)

- Square Matrices
  - Invertible (= nonsingular) iff \( \exists B, BA = AB = I_n \) \( (S \text{ singular } \Leftrightarrow \text{not invertible}) \)
  - \( A A^{-1} = A^{-1} A = I_n \), \( (A^{-1})^{-1} = A \), \( (A^T)^{-1} = (A^{-1})^T \)
  - \( (A_1 A_2 \cdots A_m)^{-1} = A_m^{-1} A_{m-1}^{-1} \cdots A_1^{-1} \)

- Matrix properties:
  - Diagonal matrix: square \( V i_{jj}, i \neq j \Rightarrow a_{ij} = 0 \)
  - Scalar matrix: diagonal \( V i_{ij}, i \neq j \Rightarrow a_{ij} = \alpha \)
  - Identity matrix: scalar \( V i_{ij}, i \neq j \Rightarrow a_{ij} = 1 \)
  - Upper triangular matrix (resp. lower triangular matrix): \( \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \) \( \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \)
  - Symmetric matrix: \( A = A^T \) \( i.e., V i_{ij}, a_{ij} = a_{ji} \)
  - Skew symmetric matrix: \( A^T = -A \) \( i.e., V i_{ij}, a_{ij} = -a_{ji} \)

MAT223 LI
LEC02 - Solving Systems of linear equations

- Row echelon form (REF)
  - all rows of zeros are at bottom of matrix
  - first nonzero entry of every row is a leading one
  - leading ones are to the right and below the previous one

- Reduced row echelon form (RREF)
  - every column contains a single leading one

- Column echelon form and reduced CEF are transpositions of those.

- Every square matrix in RREF is either \( I_n \) or has a row of zeros \([0 \ldots 0]\).

- Elementary operations
  - elementary row (col.) operations:
    - interchange two rows: \( R_i \leftrightarrow R_j \) (or cols)
    - multiply row (or col.) by nonzero \( k \): \( R_i \rightarrow kR_i \)
    - linear combinations of rows (or cols): \( R_i \rightarrow R_i + kR_j \)
  - Matrices are row (col.) equivalent if they differ by a sequence of elem. row ops.
  - A nonzero matrices, \( \exists \) equivalent matrix in REF; \( \exists ! \) equiv. matrix in RREF.
  - If the augmented matrices of two linear systems are equivalent, the systems are equivalent.

- Gaussian elimination
  - Process to reduce matrix to REF
    - forward elimination: use elementary row ops to obtain REF (or degenerate)
    - backward substitution: substitute results bottom-up
  - Gauss-Jordan elimination: use elementary row ops to obtain REF (or degenerate)
  - Linear system has \( 0 \) solutions if \( A \) matrix becomes degenerate: \( \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \)
  - \( 1 \) solution if \( A \) matrix becomes identity: \( I_n \)
  - \( \infty \) solutions if \( A \) matrix has rows of zeros: \( \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \)

- Elementary matrices
  - An elementary matrix is a matrix that differs from \( I_n \) by a single elementary row op.
  - Performing an elem. row op on \( A \) is equivalent to premultiplying \( A \) by the elementary matrix that differs from \( I_n \) by the same operation: \( B = A_{op} \iff B = E \circ A \)
  - \( A \) and \( B \) are row-equivalent iff \( \exists E_i, B = E_k \ldots E_1 A \) (with \( E_i \) elem. mat.)

- Non-singular matrices
  - Equivalent statements: \( A \) is invertible \( \iff \) \( A \) is the product of elementary matrices
    \( \iff \) \( A \) is row-equivalent to \( I_n \)
    \( \iff \) \( A x = 0 \) has a unique trivial solution
    \( \iff \) \( A x = B \) has a unique solution (per vector \( B \))

- Finding the inverse: apply Gauss-Jordan on augmented matrix with \( I_n \): \( \begin{bmatrix} A & I_n \end{bmatrix} \)
  - To obtain: \( \begin{bmatrix} I_n & A^{-1} \end{bmatrix} \)
    - Then: \( E_k \ldots E_i [A | I_n] = [I_n | A^{-1}] \)

- The inverse of an elem. matrix is obtained easily by performing the exact opposite op. to \( I_n \)
LECO3 - Vector spaces and subspaces

→ Real vector spaces

- Def: set $V$ with two operations: + vector addition: $V \times V \rightarrow V$, such that:
  1. identity elem (0 + a) = a (2) associativity (a + (b + c)) = (a + b) + c
  2. commutativity (a + b) = (b + a)
  3. negative elem (a + (-a)) = 0

→ Subspaces

- Def: subset of vectors space $V$ with same properties $+$ and $\times$.
- $W$ is a subspace of $V$ iff $W \subseteq V \land W$ is closed under + and $\times$.

A vecspc $V$, $W$ and $S$ are subspaces of $V$.

LECO4 - Span & Linear Independence

→ Span

- Def: set of linear combinations: $\text{span}\{v_1, \ldots, v_n\} = \{w \mid w = \sum a_i v_i\}$
- $S \subseteq V \Rightarrow \text{span}(S)$ is a subspace of $V$.
- span $(S) = V \Leftrightarrow S$ is a spanning set of $V$.

→ Linear Independence

- Def: set of vectors $\{v_1, \ldots, v_n\}$ is linearly independent iff $\sum a_i v_i = 0 \Leftrightarrow a_i = 0$ i.e. no vector in $\{v_1, \ldots, v_n\}$ is a linear combination of others.
- To show that property, consider the coefficient matrix $V = [v_1 \ldots v_m]$. Use Gauss-Jordan to show that $V\vec{x} = \vec{0}$ admits a unique trivial solution, if not, the vectors are linearly dependent.

LECO5 - Basis, Dimension, Rank

→ Basis

- Def: $\{v_1, \ldots, v_n\}$ is a basis for vecspc $V$ iff span $\{v_1, \ldots, v_n\} = V$ and for every $v_i$, all its values are 0 except one.

→ Dimension

- Def: dim $V$ (of vecspc) = $|B|$, where $B$ is a basis for $V$.
- $V$ is finite-dimensional iff it has a finite linearly independent set (basis).

The maximal independent subset $T$ of $S$ is a subset of $S$ such that $T$ is linearly independent and there is no other subset in $S$ that properly contains $T$.

Thm: $S \subseteq V \land \text{span } S = V \Rightarrow$ max indep. subs of $S$ is a basis for $V$.

A $S$ is a basis for $V$ $\Leftrightarrow$ $\exists 3!$ lin. combi of vecs in $S$ equal to a given vector in $V$. 


corrected
Corollaries:  \( \dim V = m \implies \forall S \subseteq V, |S| > m \implies S \text{ is lin. dep.} \)
\( \implies \forall S \subseteq V, |S| < m \implies \text{span } S \neq V \)
\( \implies \forall S \subseteq V, |S| = m \text{ and } \text{span}(S) = V \implies S \text{ is a basis for } V \)
\( \implies \forall S \subseteq V, |S| = m \text{ and } S \text{ is lin. indp.} \implies S \text{ is a basis for } V \)

Process for finding a basis for \( V \):
- Find \( S \subseteq V \), such that \( \text{span } S = V \) \( (S = \{v_1, \ldots, v_m\}) \)
- \( T \subseteq V \), such that \( T \) is a basis for \( V \) \( (T \text{ is the rank indep. subs. of } S) \)
- Use Gauss and elimination on augmented matrix of homogeneous system \([V \mid 0]\)
- Rows with a leading one are those in \( T \).

**Rank**

- The column space of \( A \) (matrix) is the span of the col. vectors of \( A \).
- The row space of \( A \) (matrix) is the span of the row vectors of \( A \).
- \( \text{Row rank}(A) = \dim(\text{rowspc } A) ; \text{col rank}(A) = \dim(\text{colspc } A) \)

**Theorem:** A row-equiv \( B \implies \text{rowspc } (\text{resp. colspc } A) = \text{rowspc } (\text{resp. colspc } B)

Process for finding a basis for \( V \):
- Repeat previous method by forming aug. mat. of hom. sys \([V_1 \cdots V_m \mid 0]\)
- Nonzero rows will form basis for rowspc of \( \text{REF}(A) \).
- Since \( A \iff \text{REF}(A) \iff \text{colspc } A \iff \text{colspc } \text{REF}(A) \), it is also a basis for rowspc.
- Note: we can apply this method to find a basis for the col space of \( A \) by using \( A^T \). This would be equiv. to using \( A \) with CEF(A).

**Rank** \( A = \text{row rank}(A) = \text{colm rank}(A) \) \( (\iff \text{ rank } A = \text{ rank } A^T) \)

**Note:** Rank \( A \) is the number of pivots in \( \text{REF}(A) \)
\( \iff \) Nonzero rows in \( \text{REF}(A) \) or \( \text{REF}(A^T) \)

**Nullity**

- The Null space (or Kernel) of a matrix \( A \) is the solution space of \( A \theta = 0 \).
- \( \text{nullity}(A) = \dim(\text{null spc } A) \)

**Theorem:** \( \text{rank-nullity} : A_{m \times n} \implies \text{rank } A + \text{nullity } A = m \) (sub of cols in \( A \))
\( \text{rank } A + \text{nullity } A^T = m \) (sub of rows in \( A \))

**LC06 - Vector Geometry**

- **Lines in \( \mathbb{R}^3 \):**
  \( P_0 = (x_0, y_0, z_0) \), \( \vec{v} = [x_0, y_0, z_0] \)
  \( P = (x, y, z) \)
  \( \text{Parametric Equation} \)
  \( \text{Distance in } \mathbb{R}^3 : d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \)

- **Norm of \( v \) in Eucl Space:** \( \|v\| = \sqrt{x_1^2 + \cdots + x_n^2} \)

- **Geometric Interpretation of Subspaces of \( \mathbb{R}^n \):**
  - Subspaces of \( \mathbb{R}^n \): \( \{0\} \) and \( \mathbb{R}^n \)
  - Subspaces of \( \mathbb{R}^2 \): \( \{0\}, \mathbb{R}^2 \) and any line passing through the origin.
  - Subspaces of \( \mathbb{R}^3 \): \( \{0\}, \mathbb{R}^3 \) and any plane through the origin.
**D Dot Product**

\[ \forall u, v \in \mathbb{R}^n, \quad u \cdot v = u^T v \]

\[ (-D) \forall u, v \in \mathbb{R}^n, \quad [u_1, u_2] [v_1, v_2]^T = u_1 v_1 + u_2 v_2 \]

- \( \vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0 \)
- \( \vec{u} \cdot \vec{v} \geq 0 \iff (\vec{u} \cdot \vec{v} = 0 \iff \vec{v} = \vec{0}) \)
- \( \text{comp}_{\vec{v}}(\vec{u}) = (\vec{u} \cdot \vec{v}) \frac{\vec{v}}{\|\vec{v}\|^2} \)
- \( \text{proj}_{\vec{v}}(\theta) = (\vec{u} \cdot \vec{v}) \frac{\vec{v}}{\|\vec{v}\|^2} \)

**LEC07 - Inner Product Spaces**

- **Def**: Vector Space \( V \) with inner product \( \langle \cdot, \cdot \rangle \), such that:
  - The inner product on \( \mathbb{R}^n \) (\( \langle \cdot, \cdot \rangle \) is the dot product)
  - **A Euclidean Space** is a finite-dimensional inner product space.

  **Cauchy-Schwarz Inequality**:
  \[ \forall u, v \in V, \quad |\langle u, v \rangle| \leq \|u\| \|v\| \]

  \( \{v_1, \ldots, v_n\} \subseteq V \) is orthogonal \iff \( \forall u \in V, \forall v \in \{v_1, \ldots, v_n\}, u \perp v \) (equivalently, \( u \cdot v = 0 \))

  \( \{v_1, \ldots, v_n\} \) is orthonormal \iff \( \{v_1, \ldots, v_n\} \) is orthogonal and \( \forall v \in V, \|v\|^2 = 1 \)

  **Thm**: \( S \) is orthogonal \implies \( S \) is lin. indep.

**D Inner Product Spaces**

- **Gram-Schmidt Process**
  - \( V \in \langle \text{Evcl. Spc.} \rangle \exists S \langle \text{ orthogonal basis for } V \rangle, \) and \( \{\text{span}(u, u_2) \}, \forall \)
  
  \[ \forall v \in V, v = \sum_{i=1}^n c_i u_i, \quad c_i = \langle v, u_i \rangle \]

  **Gram-Schmidt process provides a constructive proof for**

  \( \forall W \subseteq V, \exists S \langle \text{ orthogonal basis for } W \rangle \)

  - **Let** \( S \langle \text{ basis for } W \rangle = \{u_1, \ldots, u_n\} \). Let \( v_1 = u_1 \).
  
  - **Call** \( W_1 = \text{span} (v_1, u_2) \). Let \( v_2 \in W_1 \) such that \( v_1 \cdot v_2 = 0 \)
  
  - **Then** \( (c_1 v_1 + c_2 v_2) \cdot v_1 = 0 \implies c_1 = -\frac{v_1 \cdot v_2}{v_1 \cdot v_1} = -\frac{v_2}{v_1} \cdot v_1 \)
  
  - **Compute** \( v_2 \) and deduce \( W_1 = \text{span} (v_1, v_2) \).

  - **Repeat** the previous steps by defining: \( v_3 = u_3 - \text{proj}_{W_1} u_3 \)
  
  **Normalize the resulting basis**.
Orthogonal Complement

\[ \text{Def: For any subspace } \mathcal{W} \text{ of an IPS } V, \text{the } \text{orth. comp: } \mathcal{W}^\perp = \{ v \in V \mid \forall w \in \mathcal{W} \} \]

\[ \text{Theorem: } \forall \mathcal{W} (\text{subspace of } V), \mathcal{W}^\perp \text{ is a subspace of same } V \text{ and } \mathcal{W} \cap \mathcal{W}^\perp = \{ 0 \} \]

\[ \forall \mathcal{W}^\perp \text{ (subspace of } V \text{ (subspace of } V)\), \quad (\mathcal{W}^\perp)^\perp = \mathcal{W} \]

\[ \mathcal{W} \oplus \mathcal{W}^\perp = V \quad \text{iff } \forall v \in V, \exists w_1, w_2 \text{ such that } v = w_1 + w_2 \]

Fundamental Subspaces of a Matrix

\[ \mathbf{A} \in \mathbb{R}^{m \times m} (\text{Am} \times m \text{)}, \mathbf{A} \text{ induces } 4 \text{ fundamental subspaces: } \text{Null } \mathbf{A}, \text{ row spc } \mathbf{A}, \text{ col spc } \mathbf{A}, \text{ Null } \mathbf{A}^T \]

The Fundamental Theorem of Linear Algebra states that:

\[ \begin{align*}
\text{Row spc } \mathbf{A} &\subset \mathbb{R}^m, \text{ Null } \mathbf{A} \subset \mathbb{R}^n, \text{ Row spc } \mathbf{A} = (\text{Null } \mathbf{A})^\perp \to \text{Row spc } \mathbf{A} \oplus \text{Null } \mathbf{A} = \mathbb{R}^m \\
\text{Col spc } \mathbf{A} &\subset \mathbb{R}^m, \text{ Null } \mathbf{A}^T \subset \mathbb{R}^n, \text{ Col spc } \mathbf{A} = (\text{Null } \mathbf{A}^T)^\perp \to \text{Col spc } \mathbf{A} \oplus \text{Null } \mathbf{A}^T = \mathbb{R}^m
\end{align*} \]

LEC 08 - Linear Transformations

Def: \( \mathbf{L} : V (\text{vec spc}) \to W (\text{vec spc}) \) such that:

\[ \mathbf{L}(u + v) = \mathbf{L}(u) + \mathbf{L}(v) \quad (\forall u, v \in V) \]

\[ \mathbf{L}(cu) = c\mathbf{L}(u) \quad (\forall c \in \mathbb{R}, \forall u \in V) \]

Direct Consequences:

\[ \mathbf{L}(0) = \mathbf{L}(0) = 0 \]

Thm: Every matrix transformation \( [\mathbf{A} : u \to \mathbf{A}u] \) is a lin. trans.

Thm: \( \mathbf{L} : V \to W, \mathbf{L}' : V \to W \), \( \mathbf{L}, \mathbf{L}' \) (lin. trans.). \( S = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) (basis for \( V \))

\[ \forall \mathbf{w} \in S, \mathbf{L}(\mathbf{w}) = \mathbf{L}'(\mathbf{w}) \Rightarrow \forall \mathbf{v} \in V, \mathbf{L}(\mathbf{v}) = \mathbf{L}'(\mathbf{v}) \]

Thm: If \( V, W \) are finite dimensional, every lin. trans can be rep. by a matrix trans.

(Special case): \( \mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m \), \( \mathbf{L} \) (lin. trans.). \( \{ e_1, \ldots, e_m \} \) is the natural basis of \( \mathbb{R}^m \).

\[ \forall \mathbf{x} \in \mathbb{R}^m, \mathbf{L}(\mathbf{x}) = \mathbf{Ax} \quad (\text{where } \mathbf{A} = [\mathbf{L}(e_1), \ldots, \mathbf{L}(e_m)]) \]

Note: \( \mathbf{A} \) is unique, it is the standard matrix representing \( \mathbf{L} \).

(General case): \( \mathbf{L} : V \to W \), \( \mathbf{L} \) (lin. trans.). \( \dim V = n, \dim W = m \) (null m \& m).

\[ \forall (\mathbf{v}, S) \in V \times W, \quad \mathbf{T}(\text{ordered basis for } V) \in (\text{ord. b. for } W), \exists \mathbf{A}, \mathbf{v} \in V, [\mathbf{L}(\mathbf{v})]_S = \mathbf{A}[\mathbf{v}]. \]

Note: Basis where order of vectors is fixed.

Note: \( [\mathbf{v}]_S = [a_i] \) (where \( v = a_1 \mathbf{w}_1 + \ldots + a_m \mathbf{w}_m \) where \( S = \{ \mathbf{w}_1, \ldots, \mathbf{w}_m \} \)

is the coordinate vector of \( \mathbf{v} \) with respect to the ordered basis \( S \).

Note: \( \{ e_1, e_2, \ldots, e_n \} \) is an ordered basis.

Sorry for the mess!
Def: For any lin. trans. $L$, the kernel of $L$: $\ker L = \{v \in V | L(v) = 0 \}$

Note: $\ker (L + L') = \ker L \cap \ker L' \subseteq \ker L \cup \ker L'$

Thm: ker $L$ is a subspace of $V$.

L is one-to-one iff $\ker L = \{0\}$.

For $x, y \in V$, $L(x) = L(y) \Rightarrow x - y \in \ker L$.

Def: For any lin. trans. $L$, the range of $L$ (or image of $L$): $\text{im} L = \text{range} L = \{L(v) | v \in V\}$

Thm: $\text{im} L$ is a subspace of $W$.

$L$ is onto iff $\text{im} L = W$.

Thm: $\dim \ker L + \dim \text{im} L = \dim V$.

[$\dim V = \dim W \Rightarrow L$ is one-to-one $\iff L$ is onto$]$

Invertibility

Def: $L$ is invertible iff $\exists! L^{-1}: W \rightarrow V$, $L \circ L^{-1} = I_W$ and $L^{-1} \circ L = I_V$.

Thm: $L$ is invertible iff $L$ is one-to-one and $L$ is onto.

(Thm: $L$ is one-to-one iff $V \subseteq \text{range} L$, $S$ linearly independent $\Rightarrow L(S)$ linearly independent$)$

LEC 09 - Determinants

Def: For any nxn matrix $A$: $\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(i),i}$

Thm: $\det(A) = \det(A^T)$.

Note: $\det$ is an alternating multilinear map, thus: $\det(k_1, \ldots, k_n) = \det(k_1 e_1, \ldots, k_n e_n)$

Thm: $A$ is invertible $\iff \det(A) \neq 0$.

Thm: $A$ is singular $\iff \det(A) = 0$.

Co-factors Expandation

Def: The minor $M_{ij}$ of $A_{mn}$ for entry $a_{ij}$ is obtained from $A_{mn}$ by removing its $ith$ row and $jth$ column.

The co-factor $C_{ij}$ of $A_{mn}$ for entry $a_{ij}$ is: $C_{ij} = (-1)^{i+j} M_{ij}$.

Thm: $\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$ for any column $j$.

$= \sum_{i=1}^{n} a_{ij} C_{ij}$ for any row $i$.

Note: To manually compute determinants, reduce (keep track of $\det(A) = -\det(A^T)$) then expand.

Note: For a matrix in REF, the determinant will be the product along the diagonal! It follows that $\det(A) = 0$ iff $A$ has a row of $0s$ (iff $A$ is singular).
Def: The cofactor matrix of $A_{mxn}$ is $C_A = [c_{mi}]$ (matrix of its cofactors).

Then: $A \cdot \text{Adj}(A) = \text{Adj}(A) \cdot A = \det(A) \cdot I$


Cramer's Rule

Def: procedure for solving systems with invertible coef. matrix $A_{mxn}$.

- For $Ax = b$, since $x = A^{-1}b \iff \det(I-\lambda I_m) = \det(A) \iff x = \det(A)^{-1} \text{Adj}(A)b$

- $\therefore \forall i \in \{0..m\}$, $x_i = \det(A)^{-1} \sum_{j=1}^{m} c_{ij} b_j$ (along col.i)

- And $\sum_{j=1}^{m} c_{ij} b_j$ is the cofactor expansion of $A_i = A_{mxm}$ where col.i is replaced.

- $\therefore \forall i \in \{0..m\}$, $x_i = \det(A_i) / \det(A)$

LEC 10 - Eigenvalues & Eigenvectors

Def: $\lambda$ (eigenvalue of $A_{mxm}$) iff $\exists x \neq 0 \in \mathbb{R}^m \setminus \{0\}$ s.t. $Ax = \lambda x$ (eigenvector $A$)

Note: motivation from linear maps: eigenvectors of $T: V \rightarrow W, x \mapsto Ax$ are vectors in $V$ that their image by $T$ does not change direction.

- Characteristic Polynomial

Note: To find eigenvalues $\lambda$ of $A$: $A\mathbf{x} = \lambda \mathbf{x} \iff A - \lambda I_m \mathbf{x} = 0 \iff \det(A - \lambda I_m) = 0$

- Def: For $A_{mxm}$, $\det(A - \lambda I_m)$ is the characteristic polynomial of $A$. (its degree is $n$)

- Thm: The eigenvalues of $A_{mxm}$ are the roots of its char poly (sols of char eq.)

- Def: The eigenspaces of $A_{mxm}$ are the subspaces $E_{\lambda} = \text{null}(A - \lambda I_m)$

Note: For each eigen $\lambda$, the corresponding $E_{\lambda}$ will contain all respective eigenvectors and the $\lambda$ vec.

$\cong$ Eigenvectors characterise matrices, so, in general, $A$ and $A + cI_{mxm}$ do not hold the same.

Diagonalization

Def: $A$ is diagonalizable iff $\exists P_{mxm}, D_{mxm}, P \text{ is nonsingular}, D \text{ is diagonal}, P^{-1}AP = D$

Note: $P^{-1}AP = D \iff AP = PD \iff \{v = (c_1, c_2, \ldots, c_m) : \text{vec}(A)v = \text{vec}(D)v \}$

- $\exists$ $P$ in $\text{Mat}(m, m)$ s.t. $P$ is in $\text{Mat}(m, m)$, $P^{-1}$ is in $\text{Mat}(m, m)$, $P^{-1}AP$ is diagonal.

- Thm: $A$ is diagonalizable iff $A$ has $m$ lin indep eigenvectors that form a basis for $\mathbb{R}^m$.

Note: $D$ is a diagonal matrix with eigenvalues of $A$ along the diag (there are at least $m$)

Note: Diag. when $\dim E_{\lambda} = \text{multiplicity } \lambda$. $V \ni \text{eigens of } A$, $A$ can be diagonalized. $\cong$
LECO 1 - Algebraic Structures


- Abelian group: respects commutativity.

- Ring: \((G, +, \cdot)\) satisﬁes distributive, assc, identity.

- Commutative ring: satisﬁes commutativity (e.g., \(\mathbb{Z}\)).

- Field: satisﬁes invertibility (for non-zero).

Thus a field has 0, 1 distinguished. By closure we know that:

\[ [\text{not: } F = (F, +, \cdot)] \]

- \(\forall a \in F \) and \(a + 1 \in F\) (i.e., from any field, \(a + 1 \in F\) are automatically constructed).

- Field is not invertible, \(\mathbb{Z}: \text{not invertible}\). A commutative ring.

- Galois fields (aka, finite fields)

| Def: Field w/ non-zero characteristic \(\Rightarrow |F| < \infty\). |
| --- |
| underlying set contains zero semantics. (GARY: terms are endowed with |
| Prop: # of elements in a field is pk, p prime, |
| (\(\leq\)) For every prime p and natural k, \(F_k\) exists - |
| Two fields with same # of elements are isomorphic. |

Prime fields (i.e., \(\mathbb{F}_p \leftrightarrow 1\))

- \(\mathbb{F}_p\) is the smallest field. (by def of field: \(0 \neq 1 \Rightarrow \min |F| = 2\), it is a binary field.

- Every prime field is isomorphic to the ring of integers modulo \(p; \mathbb{Z}/p\).

Modular arithmetic reduction
\[ a \equiv b \pmod{m} \iff a - b \mid \text{div } m = q, \lambda (a - b) \mod m = 0 \text{ where } q \in \mathbb{Z} \]
\[ a = b + qm \iff \text{div: } m \mapsto p, p \text{ is the quotient of } m \text{ mod } m \模: m \rightarrow p, \text{ the remainder of } m \text{ mod } m \]
Consider the function $f_m: \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ (i.e. $f_m(x) = x \mod m$).

$\mathbb{Z}/m\mathbb{Z} = f_m(\mathbb{Z})$, i.e. $\mathbb{Z}$ is mapped to $\mathbb{Z}/m\mathbb{Z}$. The resulting equivalence class (also: congruence class, residue class) is denoted: $[a] = \{x + km | k \in \mathbb{Z}\}$.

Thus $\mathbb{Z}/m\mathbb{Z}$ is a set of integers modulo $m$:

\[ \mathbb{Z}/n = \{\ldots, -n, -1, 0, 1, 2, \ldots\} \]

$\mathbb{Z}/4 = \{[0], [1], [2], [3]\}$

And by extending $+$ and $\cdot$ to $\mathbb{Z}/n$ we get: (from the congruence ring)

$[a+b] = [a] + [b]$ & $[ab] = [a][b] \Rightarrow [a]^{-1}$ exists iff $\exists [b] \neq [0]$.

Thus $\mathbb{Z}/m\mathbb{Z}$ is also a commutative ring.

$\mathbb{Z}/m\mathbb{Z}$ respects the field axioms iff $m$ is prime (or a prime power).

Hence $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ for all $n$ prime.

Gary: Possible to have inf. field w/ finite char. e.g. mod polynomials with coeffs from finite set.

Number system digression:

$\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S}$

Vector space

*Def.* Mathematical structure that contains objects called vectors that are subjected to elements of a field through scalar multiplication consisting of a set of elements called vectors, an addition operation respecting commutative, associative, distributive, and field of vectors and the field called scalar multiplication. Respective, distributive field, distributive vector, (commutative) wrt.

Group theory digression

- homomorphism - structure-preserving map
- automorphism - self isomorphism
- isomorphism - bijective homomorphism
- endomorphism - self homomorphism
- homeomorphism - topological isomorphism
- diffeomorphism - isomorphism for smooth manifolds

Euler's alg. and its role to irreducibility in Galois fields...

Gary: By reversing Euler's alg. can find inverses...

Gary: Factor is every p-nilpotent, e.g. $F_7(9)$ is 2-nilpotent. Multiply $(p, m)$. 2-nilpotent $\Rightarrow 2x = 4$ and use symmetry $4 + 1 = 3 \times 5$
Def: sys of eqns of linear polys with coeffs in some field $\mathbb{F}$:

$$S = \{a_0 x^n, \ldots, a_k x^0 \} \rightarrow [S] = \begin{bmatrix} a_0 & \cdots & a_k \end{bmatrix} \rightarrow [M] = [K]$$

- The rows of $M$ is an upper bound on dim (solution space).
- Rows can be swapped without changing the solution space.
- Linear combos of rows can be taken, as long as nothing gets lost.
- Row ops preserve the solution space.
- (Which is the null space when the sys is homogeneous.)

- Matrix equation:

$$S = [A x = b] \quad \text{where } A \text{ is coefficient matrix}$$

The sol' of:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [\text{sol of } A x = 0]$$

- CAYE: If we look at $A$ as the function $[x] \rightarrow [b]$ then we realise that every non-homogeneous sys (i.e. $b \neq 0$) is a linear combination of solutions of the homogeneous sys $A x = 0$ (i.e. $b = 0$)

proof:

$$A x = 0 \text{ for some } x$$

$$A x + b \text{ for some } x'$$

- By linearity: $A (x + x') = A (x) + A (x') = 0 + b = b$

Interpretation: so if $x'$ is a sol to nonh, then $S x + x'$ (i.e. sol to hom) is the sol space to nonh.

- Substitutions:

Omission: elementary row ops (premultipliaction) correspond to.$^*$

Consider $S_1, S_2$ 2 lin sys s.t. $S_2$ has as many equations as $S_1$ has variables.

Then substituting the vars in $S_1$ by their expressions in $S_2$ is isomorph to matrix multiplications.

- Linear sys $\propto$ Vector spaces

The sol space to a lin sys can be parameterized in $\#$ ways.

A vector space is also parameterizable in $\#$ ways (regardless of the basis it is the same space!)

$^*$ A special kind of substitution: $A_{S_2}^{-1} R_2 \rightarrow R_2 + R_1$ (elementary row ops)

$S_1 \rightarrow [A]_{S_1} \rightarrow [I]_{S_1} = [A]_{S_2}$

E.g.

$$\begin{bmatrix} x + y = 2 \\ x + y = 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Linear sys $\propto$ Vector space
The solid space is either a point, line, plane etc.
A subspace of a vector space has those same properties.
Vector spaces will be a powerful tool to study linear systems.
Vector spaces are only characterised by a dimension - the same any set of coordinates/basis will produce isomorphic parameterisations.
Concretely, vector spaces (and subspaces) have the intrinsic properties of the solid space to homogeneous lin sys.

**Prop (Ax = 0):**
\[ x_1, x_2 \in \text{Sol} \]
\[ \Rightarrow x_1 + x_2 \in \text{Sol} \]

There is always a trivial sol\[ (0) \]
\[ \Rightarrow \text{the 0 vector } [0] \]

\[ x_1 \in \text{Sol} \]
\[ \Rightarrow \alpha x_1 \in \text{Sol} \]
\[ (\alpha \in \mathbb{R}^F) \]

Row space & matrices

\[ \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \times \begin{bmatrix} x_5 & x_6 & x_7 & x_8 \end{bmatrix} \]

columns lie in cols of matrix \[ \Leftrightarrow \text{if } LT \text{ is } LT, \text{ then cols lie in the range} \]

codomain = range \[ \Rightarrow T \text{ is } 1-1 \]

c-A-R-V: think of cols already in the range \[ (\in \text{codomain}) \]

\[ \Rightarrow \text{computing the range is } P \text{, constructing a subspace in the codomain whose dimension is } \]

**Axioms of Vec Spc**

- Vector addition is binary (ie \[ V \times V \rightarrow V \])
- Binary ops are closed.
- \[ \Rightarrow x_1, x_2 \in V \Rightarrow x_1 + x_2 \in V \]
- \[ \Rightarrow 0 \in V \]
- Scalar mult is binary \[ (\Rightarrow \text{closed}) \]

\[ x_1, x_2 \in \text{Sol} \]
\[ \Rightarrow x_1 + x_2 \in \text{Sol} \]
LECO3 - Change of bases & coordinates

- Possible subspaces of $V_2$

  e.g. $V_2 = \{ [y] \mid x \in \mathbb{R} \}$

  \[ \begin{align*}
  &\text{Matrix equiv. to plane } V_2 \\
  &\text{1st eq. } \{ 103 \} \Rightarrow \text{a line in } V_2 \\
  &\text{2nd eq. } \{ 010 \} \Rightarrow \text{a point in } V_2
  \end{align*} \]

  They are the same subspace (i.e. $V_2$) but different subspaces of $V_2$.

- Two perspectives on subspace

  - As span of a basis
  - As space of sol to lin sys

  e.g. Let $W \subseteq V_3$ s.t. $W = \{ [x] \mid x - y - z = 0 \}$

  \[ \begin{bmatrix}
  1 & 0 & 1 \\
  1 & 1 & 0 \\
  2 & 1 & 1
  \end{bmatrix} \Rightarrow \begin{bmatrix}
  x &=& 1 - x - z \\
  y &=& -z \\
  z &=& z
  \end{bmatrix} \]

  \[ W = \text{span } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \text{span } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

  \[ W = \text{span } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \text{span } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \]

  We need a sys $W / 2$ equiv. s.t. 1 free variable (parameter of sol spec)

  \[ W = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ x } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in W \text{ thus: } x + 2z = 0, \]

  \[ \begin{align*}
  &x + 2z = 0 \\
  &-x + 2z = 0
  \end{align*} \]

  \[ W = \text{Sol } \begin{bmatrix} 100 \\ 010 \\ 100 \end{bmatrix} \]

- Changing coordinates (will be revisited as it is a mess!)

  Let $\alpha = \{ u_1, u_2 \}$ be a basis for $V_2$. An element of $V_2$ is written under that basis.

  \[ \begin{align*}
  &\text{Change of basis}\ \text{matrix: } P_{\alpha} \text{ from } \alpha \text{ to } \beta \text{ is } \\
  \end{align*} \]

  Let $\beta = \{ v_1, v_2 \}$ s.t. $\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

  \[ \begin{align*}
  &x_1 v_1 + x_2 v_2 \quad \text{coordinates for the basis } \\
  \end{align*} \]

  \[ \begin{align*}
  &x' v_1' + x_2 v_2' \quad \text{for the basis } \\
  \end{align*} \]

  Let $\beta = \{ v_1, v_2 \}$ s.t. $\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

  \[ \begin{align*}
  &x_1 v_1 + x_2 v_2 \quad \text{coordinates for the basis } \\
  \end{align*} \]

  \[ \begin{align*}
  &x' v_1' + x_2 v_2' \quad \text{for the basis } \\
  \end{align*} \]

  Thus $v = x_1 v_1 + x_2 v_2$, so hand die
In this example:

1. A real vector space is a vector space over \( \mathbb{R} \).

2. Consider \( V_3 \) over \( \mathbb{R} \) (aka \( \mathbb{R}^3 \))—does not need to have the 'Euclidean' orthonormal basis.

3. Consider the linear map \( T: \mathbb{R}^3 \to \mathbb{R}^3 \)

   **Cases for kernel shape:**
   - 1) \( T = \text{Im} \Rightarrow \) all cols are lin. indep. \( \implies \) range = codomain
   - 2) \( T = 0_m \Rightarrow \text{range} = 0 \)

   For 2) & 3) we have the fact that \( T \) brings at least 2 things in domain to 1 thing in range (which is not 0).

   Thus, this point in the range, call it \( b \), is an image of a shape in the domain which is the kernel translated.

4. Recall that everything in the domain that points to a single point in the range has the same shape as the kernel of \( T \)

   What goes to 0 under \( T \) is the nullspace of \( T \).

2) Hence if the kernel is a line, everything that points to a single point is a line as the domain can be filled with line.

   \[ \begin{align*}
   \text{kernel} & \quad \overset{T}{\rightarrow} \quad \text{range} \\
   \text{by compressing the lines on themselves} & \quad \overset{T}{\rightarrow} \quad \text{we get a plane}
   \end{align*} \]

3) If kernel is plane, we can fill the domain by stacking planes.

   \[ \begin{align*}
   \text{kernel} & \quad \overset{T}{\rightarrow} \quad \text{range} \\
   \text{by compressing the planes on themselves} & \quad \overset{T}{\rightarrow} \quad \text{we get a line}
   \end{align*} \]

- i.e. The way the kernel is stacked is the range.

- Another look at row ops: They preserve the kernel (in domain)
  - They perform \( A \) on the range:

   \[ \begin{align*}
   (A(x)) & \rightarrow \text{a pot \( x \) in the range} \\
   \text{B changes the shape of the range of} \ A & \rightarrow \text{the range of} \ A
   \end{align*} \]

And this only works for homogeneous sys

 bc. \( \vec{0} \) is the fixed point

- All other sols to non-hom can be derived from hom
Direct sum notation:
- $V$ is the direct sum of subspaces $W_1, W_2, \ldots, W_m$ if $W_1 \cap W_2 \cap \cdots \cap W_m = \{0\}$ and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_m$.
- Equivalently, $V = W_1 \oplus W_2 \oplus \cdots \oplus W_m$ if bases for $(W_i)_{i=1}^m$ can be combined to form a basis for $V$.
- Preserves vector space axioms, e.g., $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$.

Dimension theorem:
- $\text{null}(A) = \dim \text{null } T$.
- $\text{rank } A = \dim \text{Im } T$.
- $\text{rank } A + \dim \text{null } A = \dim V$, \hspace{1cm} $(\# \text{ of lin. indep. cols. } = \# \text{ of lin. indep. rows})$

Recall from Linear Algebra:
- $\text{row \ spec } A \oplus \text{null } A = V$.
- $\text{rank } A + \dim \text{null } A = m$.
- $\text{row \ spec } A \oplus \dim \text{null } A^T = W$.
- $\text{rank } A + \dim \text{null } A^T = m$.

The range (performs a LT):
- We can't use $A$ as a basis for the original matrix, but we can identify which cols were originally responsible for lin. indep. Thus, $\text{basis}$ is a basis for cols spec.

Row: all that's left is the dimension theorem.

CARY + ROW: pre/post are confusing! talk about left null on right null.

LECONS: similarity
- We saw that we only need to study homogenous systems as we need only to translate (no piv) the sol/spec (aka null spec).
- In the matrix representation of the system, we noticed that the linear system (rows) can be viewed as a vector space, but none importantly, that the col vecs lie in another vec spec, that's 'reduced' to the first.
- And thus a matrix, viewed as a LT, is a link between the domain (rows) and the codomain (cols).
Conjugacy is not $R$, $C$, or $RC$ equiv. It's equiv. to chg of coord.

Rowe: Similarity for $LT$s is $RC$-equiv.

Change of coordinates is a bijection.

Another note from Rowe: Rational Canonical Form Theory & spectral theorem are $\otimes$ subtle, caprine behaviors w/ $\neq$ reference.

Note from Rowe: In undergrad, I also conflated LTs and LT$s$ thus viewing conjugacy (for $LT$s) as a special case of similarity for square matrices.

But I realized there is a clear distinction between $LT$s and $LT$s; $\mathbf{LT} s$ obey their own structure, $\mathbf{LT}s$ and thus should be distinguished accordingly.

\[ \text{Left mult by} \]
\[ \text{GARY: notation is } L_A \]
\[ \text{GARY: in the platonistic sense, } [T]_A \text{ is enough to fully describe } T. \text{ That is, how } T \text{ changes a } \alpha \text{ fixed basis of } V. \text{ But } [T]_X \text{ is necessary to instanstand } T \text{ (represent it) as a matrix.} \]
\[ \text{(Alex), and } T \text{ as its own is pure, pure platonistic. Just as a basisless vector space it exists in the platonistic world of Abstraction. Mathematics is about formalising these intuitional concepts, meta-mathematics.}\]

So keep in mind that $\ker T = \{ v \in V : T(v) = 0 \}$, and as soon as we instanstand it for some bases $\alpha$ and $\beta$, we get:

$$\ker T = \ker [T]_\alpha$$

Thus $\forall v \in V, [T(v)]_\beta = [T]_\alpha [v]_\alpha$.

If we represent the solutions to the homogeneous underlying system, then $\ker T$ can be represented as any element in the class of matrices $R$-equiv to $[T]_\alpha$, i.e.:

$$\ker T = \ker [T]_\alpha = \ker [T]_\beta$$
Call this class $\text{REG}_{\aleph} = \{ M \mid M \text{ is r-\text{eq to } } [T]_B^B \}$.

Then any element of $\text{REG}_{\aleph}$ can be written as: $M = Q [T]_{B_B}^B$, where $Q$ is invertible.

Geometrically, this corresponds to changing the way we 'measure' the codomain, i.e., its basis. Thus, if we take a new basis $B'$ in terms of $B$ (i.e., $B' = \alpha_1 B_1 + \alpha_2 B_2$, ..., and take the corresponding change of basis matrix $B' = [aij]$, we can obtain the change of coordinate matrix $Q' = (P'_{B_B})^T$ (see "proof" on page 3).

Putting it all together, we get $[T]_{B_B}^B = Q [T]_{B_B}^B$ nonplactic.

Similarly, if we care about the range, preserving our view of the range (that is, we care about the codomain), we get the counterpart set $\{ M \mid M \text{ r-\text{eq to } } [T]_B^B \}$, and for any new basis $B'$ in the domain, the rule is $[T]_{B_B}^B = [T]_{B_B}^B Q_{B_B}$. (Notice in this case, as it is right multiplication, that we need the inverse of $Q_{B_B} = (P_{B_B})^T$, i.e., $Q_{B_B} = (P_{B_B})^{-1}$.)

Another view on these 'change of bases matrices (on their transposes for coordinates)' is presented here:

- we renoticed that we can view left multiplication as performing an $LT$ on the range of $a LT$. (Same for right $LT$, null, and kernel).
- we want our 'change of coordinate matrices' to be invertible.
- we can view them as instantiations of the identity basis form! That is, we can view our equivalence class $\text{REG}_{\aleph}$ (same for $\text{REG}_{\aleph}$) as the set $\{ Q [T]_{B_B}^B \mid Q = [T]_{B_B}^B \text{ for some } B'$, a basis for $W$).

Long awaited digression: the platonic world of Abstractness for $LT$s.

Meta-disjunction:
- in math, we grab stuff from universe and visit.
- in math, we study the ways to grab and crystallise in a basis.
- crucial theorems:

Finally, at this level $[T]_{B_B}^B$ can be represented physically, as a matrix.

- in other words, when we left multiply by an invertible matrix (size for right), we perform an isomorphism on the range.

Def 2 matrices $A, B$ of the same size are similar iff $T, Q$ invertible, such that $A = P B Q$.

\[ [T]_{B_B}^B \]
Property: Any matrix in an equivalence class defined by similarity shares the same RRCEF.

Proof: Dimension theorem:

\[ A \in \text{RRCEF} \implies \text{RRCEF}(A) = \text{RRCEF}(B) = \text{RRCEF}(C) \]

- Rowe's all that remains is the dimension formula.

- Conjugacy

Can be seen as a special case of similarity, but better seen as the counterpart of similarity for LOs.

\[ L : V \to W \quad \text{(linear mapping between vec spaces)} \]

Endomorphism (finishing a vec space)

Def. Two matrices \( A, B \) are conjugate iff \( A = Q^{-1} B Q \)

- Hence \( A = L B \) (the platonic underlying LOs are the same)

Proof: Let \( B = \{ \beta_i \} \) be bases for \( V \).

\[ A(\nu) = [L]_{\alpha}^\nu \cdot [V]_{\alpha} = A \cdot [V]_{\alpha} \quad (\text{LO are basisless}) \]

\[ = Q B (Q^{-1} [V]_{\beta}) = QB [V]_{\beta} \quad (Q^{-1} = Q_{\alpha}^\beta) \]

\[ = Q [L_B]_{\beta} = [L_B(Q)]_{\alpha} = L_b(\nu) \quad (Q = Q_{\alpha}^\beta) \]

(Alex) Think of \( Q A Q^{-1} \) as: A changing the basis of the domain, but we want to change both in the same way.

- The conjugacy class of an LO captures all possible ways to view this LO (in a single basis).

The final deal about "change of basis"

Express \( \nu' \) in terms of \( \nu \)

\[ \nu' \overset{\nu}{\mapsto} \nu \]

Knowledge of \( \nu \) gives us:

(\( \nu \) basis)

(\( \nu' \) basis)

(\( \nu \) coord)

(\( \nu' \) coord)

If we know \( \nu' \), we cannot say anything about \( \nu \).

\[ (\nabla^\nu) T = \Delta^\nu \]

\[ (\nabla^\nu') = \Delta'^\nu \]

Cases:

\[ [T]_{\alpha}^\nu \quad [T]_{\alpha}^{\nu'} \]

\[ [T]_{\alpha}^\nu = \Delta^\nu \quad [T]_{\alpha}^{\nu'} = \Delta'^{\nu'} \]

Coord.:

\[ \Delta^\nu \leftrightarrow \Delta'^{\nu'} \]

\[ \Delta^\nu \leftrightarrow \Delta'^{\nu'} \]

\[ [T]_{\alpha}^\nu = \Delta^\nu \quad [T]_{\alpha}^{\nu'} = \Delta'^{\nu'} \]
The complex field

- Let \( \mathbb{C} = \mathbb{R}^2 \) over \( \mathbb{R} \), with complex null defined as the origin.

- The space of \( 2 \times 2 \) matrices over \( \mathbb{R} \): \( \{ M \in \mathbb{R}^{2 \times 2} \mid M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \text{ for } a, b \in \mathbb{R} \} \)

- Corresponds to rotation and scaling (vectors are not rotated)

- E.g. \( i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rightarrow x(a+ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \)

- Notice that \( A^T A = D \)

- Co the behaviour of \( 2 \times 2 \) matrices encapsulates in \( A^T A = D \),

  leads to the fundamental theorem of Algebra

  (i.e. \( \mathbb{C} \) is algebraically closed)

The conjugate transpose

- Consider symmetry for real matrices i.e. A symmetric if \( A = A^T \)

- Symmetric matrices over \( \mathbb{R} \) have 1 axes of symmetry (R^d)

  \[ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

  - Extend the concept to complex matrices, they will have 2 axes (R^2)

  \[ \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \]

  - Intuitively, it makes perfect sense to generalise sym.

  - To get hirnily (sym. becomes special case for reals)
Def the conjugate transpose of a matrix $A$ is $A^* = (\bar{A})^T$.

$\neg\forall$ Again, this is the complex counterpart of transpose for reals (dim $\mathbb{R} = 1$) (dim $\mathbb{C} = 2$)

$\Rightarrow$ A matrix $A$ is Hermitian iff $A = A^*$

$\neg\forall$ Notice that the behaviour of these matrices is: $A + A^* = \mathbb{R}$ (one real)

Wow! Now what does this behavior give us? (The spectral Theorem)

1+2 $\neg\forall$ Reference to intuition: apply 2 (the fundamental theorem of Algebra) to characteristic poly of Hermitian matrices $A$.

$\Rightarrow$ The Spectral Theorem

$M$ is Hermitian iff all of $M$'s eigenvalues are real $\iff M$ is diagonalizable.

Def: $M$ is positive definite iff all of its eigenvalues are positive $\iff M$ is diagonalizable.

M is positive semi-definite iff $M^* M$ is positive semi-definite.

---

Digression: where do the rows of a matrix line?

(Alex) They must be in some space kinda constraining the rows in $V$?

GARY + ROSE: They line in the dual space:

$$\forall \alpha, \beta \in V^*, \exists \gamma \in V: \alpha(\gamma) = \beta(\gamma)$$

Here is a bijection from $V \leftrightarrow V^*$

$B: V \rightarrow V^*$

$\Rightarrow$ $b_v : V \rightarrow \mathbb{F}$

(Alex) And it goes without saying that $b_v$(span $v$) = $\mathbb{F}$.

---

GARY's example for appreciating plactic's LTs.

Consider $T_1$, $T_2$ LT's from $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ $\forall$ bases $\alpha$ for $\mathbb{R}^3$, $\beta$ for $\mathbb{R}^2$.

$T_1(\alpha_1) = \beta_0$
$T_1(\alpha_2) = \beta_1$
$T_1(\alpha_3) = \beta_1$

$T_2(\alpha_1) = \beta_0$
$T_2(\alpha_2) = 0$
$T_2(\alpha_3) = \beta_1$

$\Rightarrow$ The dimension of span $\alpha$, span $\beta$, and change of basis.

T1 projects span $\alpha_1$, $\alpha_2$ and dimension (span $\beta_0$) whereas $T_2$ collapses an entire span to the 0 vector.

However $T_1$, $K$ $T_2$ are the same plactically? This can be seen by changing the base direct to $\alpha_1$, $\alpha_2$ (i.e. the (improbable) col. op.)
LEC07 - Generalised Eigen spaces

- The determinant

\[ \text{(As seen in the previous example, LTs exist regardless of their bases.) For LOs, that are studied with a single basis (domain = codomain), we can measure how much the LO distorts the oriented area/volume given a base basis.}\]

- The trace

\[ \text{Def: } \text{tr}(A) = \sum_i A_{ii} \quad \text{(i.e. sum of elements of main diagonal)} \]

\[ \text{Thm: } \text{tr}(AB) = \text{tr}(BA) \]

\[ \text{Hence: } \text{tr}(A^{-1}A) = \text{tr}(I) \]

- The Characteristic Polynomial

\[ \text{Def: } c_T(x) = \det(T - xI) \quad \Rightarrow \text{polynomial over } \mathbb{F}, \text{ with leading coef } (-1)^n \]

\[ \text{Rowe: if } A \text{ is a poly, you can plug in matrices or LOs up to } \mathbb{F}_n \text{ i.e. } c_T(\mathbb{F}) \subseteq \mathbb{F} \]

\[ \text{Thm: } \forall T = LO, c_T(T) = 0_n \text{ (in finite matrices), i.e. } T \text{ is annihilated by its char poly.} \]

- Def: \( m_T(x) \) is the smallest degree poly \( m_T(x) \) that has leading coef 1 and annihilates \( T \text{ (aka monic)} \)

\[ \Rightarrow m_T(x) \text{ has same roots as } c_T(x) \]

\[ \Rightarrow m_T(x) \text{ divides } c_T(x) \]

\[ \text{Thm: If } c_T(x) \text{ has a root } \lambda \text{ in } \mathbb{F}, \exists \lambda \text{-eigvec of } T \]

\[ \text{proof: } c_T(\lambda) = 0 \Leftrightarrow \det(T - \lambda I) = 0 \Leftrightarrow T - \lambda I \text{ is not invertible (i.e., not an isomorphism).} \]

\[ \Rightarrow \text{dim ker}(T - \lambda I) > 1 \text{ (not an isomorphism) i.e. } \exists \lambda \text{-eigvec of } T \]

\[ \Rightarrow \exists \text{ a nonzero vector } \mathbf{v} \text{ s.t. } (T - \lambda I)(\mathbf{v}) = 0_n \text{ i.e. } \mathbf{v} \text{ is a } \lambda \text{-eigvec.} \]

- Def: The previous proof showed \( v \) where the \( \lambda \)-eigvec live, i.e.

\[ E_\lambda = \text{ker}(T - \lambda I) \text{ is the } \lambda \text{-eigenspace of } T \text{ iff } E_\lambda \neq 0_n \text{ (iff } \lambda \text{ is a root).} \]

- Irreducibility

- Def: A polynomial is irreducible over a field \( \mathbb{F} \) if it cannot be factored.

\[ \Rightarrow \text{Over } \mathbb{C}, \text{ only linear poly is irreducible (Fund. Theorem of Algebra)} \]

\[ \Rightarrow \text{Over } \mathbb{R}, \text{ linear poly and quadratic poly w/ no real root are irreducible.} \]

- Multiplicities of eigenvalues

Algebraic: exponent of eigen \( \lambda \) in corresponding factor of char poly, i.e. \( (x - \lambda)^k \)

Geometric: \( \dim E_\lambda \)
Def. Given $\lambda$, an eigenvalue of $T$, the generalized eigenspace is $K_{\lambda}^g = \{ v \in V \mid (T - \lambda I)^k v = 0 \}$

Thm. For a given $\lambda$, eigenvalue of $T$, dim $K_{\lambda}^g = a_\lambda$ (algebraic multiplicity) subspace.

$\Rightarrow$ For an eigenvalue $\lambda$, $\text{geom}(\lambda) = \text{alg}(\lambda) \geq 1 \Rightarrow E_{\lambda} \subseteq K_{\lambda}^g$

- Cyclic subspaces

Def. $K \subseteq V$ is $T$-invariant iff $T(K) \subseteq K$

Def. $K \subseteq V$ is $T$-cyclic iff $K$ is $T$-invariant & $\exists k \in \mathbb{N}, \exists v \in V, K = \text{span}(v, \ldots, T^k v)$

Thm. For $\lambda$, eigenvalue of $T$, $K_{\lambda}^g$ is $T$-invariant (also $(T - \lambda I)$-invariant) by definition.

Def. The cyclic subspace generated by $v$ (for LOT) is $C(v) = \text{span}(v, \ldots, T^k v)$ such that $\{v, \ldots, T^k v\}$ is lin indep and $\{v, \ldots, T^k v\} \cup \{T^k v, \ldots\}$ does not.

$\Rightarrow$ it is a $T$-invariant subspace.

- Annihilation subspaces

Def. $K_{\lambda}^g = \{ v \in V \mid p(T)v = 0 \}$

Thm. $C(v)$ is an annihilation subspace for $p(x)$.

proof. By def., $T^k v = \sum a_i T^i v$

$\Rightarrow T^k v = \sum a_i T^i v = 0 \Rightarrow (T^{k+1} - E)(v) = 0$

Let $p(x) =$ substitute $x$ for $T$ in $\sum a_i T^i v$ (eq substitute $T$ by $x$)

Then $C(v) = K_{\lambda}^g$.

LEC08 - Rational Canonical Form Theory

- Diagonalisation

$T$ is diagonalizable iff $C_T(x)$ factors completely over $\mathbb{F}$ and for every eigenvalue $\lambda$, $\text{alg}(\lambda) = \text{geom}(\lambda)$

$\Rightarrow$ if $C_T(x)$ factors into distinct linear factors over $\mathbb{F}$, then $T$ is diagonalizable.

$T$ is diagonalizable iff $\forall \lambda : \text{eigen of } T, E_{\lambda} = K_{\lambda}^g$

- Jordan Canonical Form & Rational Canonical Form

Def. $N : \mathbb{C}^n$ is nilpotent iff $\exists k \in \mathbb{N}^*$, $N^k = 0$

Thus $N$ is nilpotent if $m_N(x) = x^k$ for some $k$

iff $C_N(x) = (-1)^{k-1} x^k = (0-x)^k$

iff $0$ is $N$'s only eigenvalue.

iff $\forall \lambda \in \mathbb{C} \setminus \mathbb{N}^*, \text{tr}(N^k) = 0$. 

How nilpotents work: (read from right to left)

```
Ker N
```

```
N : V → V
```

```
V / Ker N
```

```
N is 1-1 on V / Ker N
```

"which goes to Ker N under N"

pull back

```
(V / Ker N) → N
```

Modeling out:

```
(V_1 < V_2), V_1 ⊕ V_2 / V_1 ≈ V_2
```

```
V_2 Mod V_1
```

So a nilpotent 'decomposes' its domain in such a way...

E.g. V = V_5, with basis (x_1, x_2, x_3, x_4, x_5)

```
N : N x_3 = x_2, N x_5 = x_4
N x_1 = 0, N x_2 = x_1, N x_4 = 0
```

```
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
```

This was a simple example. In general:

1. To get the superdiagonal, need to find the image of the 'pulled back' basis of a cycle basis.

There are thus a small number of nilpotents, this number captures how many applications of N it takes to annihilate the vector (this is the exponent of the minimal polynomial) and how big are its pulled back kernels. This number is hence simply the number of operations of an - for e.g. c_0(x) = x^3

As mentioned informally, to get the canonical form of a nilpotent, we want a 'cycle basis': from previous example:

```
ROWE : define kernel shah's as follows: (dim Ker N, dim Ker N^2, ..., dim Ker N^n)
```

The jumps in kernel can be used to determine the operator's cycle type (i.e. which partitions of n it corresponds to).

```
\text{Ker shah} = (S, S, S, 1, 1) \iff \text{cycle type} : [3, 2] \text{ (or } [2, 3])
```

Note: the greatest # in the cycle type corresponds to the exponent of the minimal poly, it is the size of the largest cycle basis.

So nilpotents are a great way to partition a vector, and using cycle bases gives a nice canonical rep.
Given a nilpotent $N^3$, we can determine to which canonical form it is conjugate, (i.e. which plaeonic nilpotent it is) as such:

- $N: V_3 \to V_3$. (Just like previous example, except with a more arbitrary computation method.)
- Compute kernel and nullity.
- Find a vector $v_1 \in \ker N^3 \setminus \ker N^2$. This gives the first cycle basis $(v_1)$.
- Take the second (2-cycle) one, repeat the same method, but make some $C(v_2) \cup C(v_1)$ as independent.
- The canonical basis for $N$ of $V_3$ is thus $B = C(v_1) \cup C(v_2)$.

$$\Rightarrow [N]_B = \Delta^Y [N]_B \Delta^Y,$$
where $[N]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Def:** The Jordan canonical form (JCF) of a $T \in T$ is a special matrix representation of $T$ that is possible when $C_T(x)$ factors completely over $F$.

- Algebraic multiplicity need not equal geometric multiplicity.
- $\mathcal{E}_\lambda \leq K^\lambda, \ K^\lambda/\mathcal{E}_\lambda$ need not equal $0$.
- We can decompose $V$ into $T$-invariant subspaces. Every $V$ can be decomposed trivially into improper $T$-inv subsp: $V \leq 0$. But none interestingly can be decomposed into proper $T$-inv subsp. (e.g. when $T$'s char poly factors completely over $F$.) (In which case, those $T$-inv subsp correspond to $K_2: \lambda$'s.)
- Notice that we can define nilpotents for each generalised eigenspc: as such, $K_2: \lambda = (\lambda - \lambda: I)$. This will allow us to construct cycle bases as previously shown.

$$\Rightarrow$$

- Those bases can be unoriented to get a Jordan basis of $T$.
- This nilpotent analysis of $T$'s generalised eigenspace will construct the Jordan block of $K_2: \lambda$ is formed as:

$$\begin{bmatrix} \begin{bmatrix} [T - \lambda: I]_B \\ \vdots \end{bmatrix} \\ \mathcal{E}_\lambda \end{bmatrix}_B = \begin{bmatrix} \mathcal{E}_\lambda \\ \mathcal{E}_\lambda \end{bmatrix}_B$$

**Def:** The Rational Canonical Form (a.k.a. Frobenius Normal Form $\sim$) is a special matrix rep. of LO $T$, that is unique (up to permutation). Thus every plaeonic $T$ is 2-tyclic.

Then $V = \bigoplus K_2: \lambda$.

**Def:** Suppose $K$ is cyclic. Let $V = K \oplus \cdots$.

The companion matrix $A_p(x)$ of $p(x) = x^d + \cdots + a_1 x + a_0$ is a polynomial such that $A_p(x) \neq 0$.

$$A_p(0) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{d-1} \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 \end{bmatrix}$$