Manipulation rules

identity	$P\land(Q\lor\neg Q)\Leftrightarrow P$	$P\lor(Q\land\neg Q)\Leftrightarrow P$
idempotency	$P\land P\Leftrightarrow P$	$P\lor P\Leftrightarrow P$
commutativity	$P\land Q\Leftrightarrow Q\land P$	$P\lor Q\Leftrightarrow Q\lor P$
associativity	$(P\land Q)\land R\Leftrightarrow P\land(Q\land R)$	$(P\lor Q)\lor R\Leftrightarrow P\lor(Q\lor R)$
distributivity	$P\land(Q\lor R)\Leftrightarrow(P\land Q)\lor(P\land R)$	$P\lor(Q\land R)\Leftrightarrow(P\lor Q)\land(P\lor R)$
De Morgan	$\neg(P\land Q)\Leftrightarrow\neg P\lor\neg Q$	$\neg(P\lor Q)\Leftrightarrow\neg P\land\neg Q$

→ implication $P\Rightarrow Q\Leftrightarrow\neg P\lor Q$ $\qquad\neg(P\Rightarrow Q)\Leftrightarrow P\land\neg Q$

→ contrapositive $P\Rightarrow Q\Leftrightarrow\neg Q\Rightarrow\neg P$

→ equivalence $(P\Leftrightarrow Q)\Leftrightarrow(P\Rightarrow Q)\land(Q\Rightarrow P)$ $\qquad\neg(P\Leftrightarrow Q)\Leftrightarrow\neg(P\Rightarrow Q)\lor\neg(Q\Rightarrow P)$

→ double neg° $\neg\neg P\Leftrightarrow P$ → absorption $P\lor(P\land Q)\Leftrightarrow P\land(P\lor Q)\Leftrightarrow P$

→ quantifiers

→ negation $\neg(\forall x, P(x))\Leftrightarrow\exists x, \neg P(x)$

$\neg(\exists x, P(x))\Leftrightarrow\forall x, \neg P(x)$

→ distribution $\forall x, P(x)\land Q(x)\Leftrightarrow[\forall x, P(x)]\land[\forall x, Q(x)]$

$\exists x, P(x)\lor Q(x)\Leftrightarrow[\exists x, P(x)]\lor[\exists x, Q(x)]$

Proofs

→ Direct : Assume $x\in D$ ... Then, $\forall x\in D$ ...

→ Indirect: Assume $x\in D$, Assume $\neg Q(x)$ ... Then $\neg P(x)$ ... Then $\forall x\in D, P\Rightarrow Q$

→ Contradiction : For a contradiction assume $\neg P$ ... Contradiction: P is true

→ Proof by cases: To prove $(A\lor B)\Rightarrow C$, case 1, assume A ... case 2, assume B ... then C

→ Proof by mat. induc°: Part 1, prove P_0 ... Part 2, prove $P_n\Rightarrow P_{n+1}$ ... Part 3, conclusion

Inference rules

→ Introduction

[∀I]
Assume $a\in D$ ... $P(a)$
→ $\forall x\in D, P(x)$

[∃I]
$P(a)\land a\in D$
→ $\exists x\in D, P(x)$

[∧I]
A, B
→ $A\land B$

[∨I]
A
→ $A\lor B$, $A\lor\neg A$

[¬I]
Assume A ... contradic°
→ $\neg A$

[⇔I]
$A\Rightarrow B$, $B\Rightarrow A$
→ $A\Leftrightarrow B$

[⇒I]
. direct: Assume A ... B
→ $A\Rightarrow B$
. indirect. Assume $\neg B$ ... $\neg A$
→ $A\Rightarrow B$

→ Elimination

[∀E]
$\forall x\in D, P(x)\land a\in D$
→ $P(a)$

[∃E]
$\exists x\in D, P(x)$
→ Let $a\in D$ such that...

[∧E]
$A\land B$
→ A, B

[∨E]
$A\lor B$, $\neg A$
→ B

[¬E]
$\neg\neg A$ → A
$A, \neg A$ → contradic°

[⇔E]
$A\Leftrightarrow B$
→ $A\Rightarrow B$, $B\Rightarrow A$

[⇒E]
. Modus Ponens :
$A\Rightarrow B$, A → B
. Modus Tollens :
$A\Rightarrow B$, $\neg B$ → $\neg A$

Asymptotic Notation

$O(f) = \{ g : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq c f(n) \}$

$\Omega(f) = \{ g : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \geq c f(n) \}$

$\Theta(f) = \{ g : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists (c_1, c_2) \in \mathbb{R}^{+2}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n) \}$

→ properties: (1) $f \in O(f)$ (2) $g \in \Omega(f) \Leftrightarrow f \in O(g)$

(3) $f \in O(g) \wedge g \in O(h) \Rightarrow f \in O(h)$

(4) $g \in O(f) \wedge g \in \Omega(f) \Leftrightarrow g \in \Theta(f)$

Countability

$$\text{bijective} = \begin{cases} \text{injective (one-to-one)}: \forall (a_1, a_2) \in A^2,\ f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \\ \text{surjective (onto)}: \forall b \in B, \exists a \in A, f(a) = b \end{cases}$$

S is countable $\Leftrightarrow \exists f : S \to \mathbb{N} \wedge f$ is one-to-one
$\Leftrightarrow \exists f : \mathbb{N} \to S \wedge f$ is onto

Diagonalizaº

→ Cantor :

$f(0): 0.0000\ldots$
$f(1): 1.1111\ldots$
$f(2): 0.1010\ldots$
$f(3): 1.0101\ldots$
$\ldots$

→ take opposite, i.e. 1.011...
this nb will never occur, random info vs infinity : beyond scope of $\mathbb{N}$-countability