CSC 373 - Algorithm Design, Analysis, and Complexity **Summer 2016**

Lalla Mouatadid

LP Rounding - An Example

Consider the following variant of vertex cover:

Weighted Vertex Cover with Edge Penalties
Input:A graph $G(V, E, w, c)$ where $w : V \to \mathbb{R}^+$, and $c : E \to \mathbb{R}^+$.
Problem Return a subset of vertices $S \subseteq V$ such that

$$\sum_{v \in S} w(v) + \sum_{\substack{(u,v) \in E \\ u,v \notin S}} c(u, v)$$

is minimized.

Notice that this is not a "real" vertex cover: We pay for vertices we collect, and we still want to cover edges, but if we decide to leave an edge e uncovered, we also pay a penalty cost $c(e)$ for it.

In this lecture, we will formulate the above problem as an $0, 1$ IP, and use LP rounding to come up with the best approximation ratio for the problem. In particular, we will first attempt a naive rounding, and then see how we can be slightly more clever to come up with a better approximation ratio.

First we formulate the problem as a $0, 1$ IP. To this end, we introduce a boolean variable x_i for every vertex $i \in [n]$, such that $x_i = 1$ if $i \in S$, and 0 otherwise. Similarly, we introduce another boolean variable e_{ij} for every edge $(i, j) \in E$, such that $e_{ij} = 1$ iff edge (i, j) is covered.

$$\begin{aligned} \text{minimize } & \sum_{i=1}^{n} w(i)x_i + \sum_{\substack{(i,j) \in E \\ i,j \notin S}} c(i, j)(1 - e_{ij}) \\ \text{s.t. } & x_i + x_j \geq e_{ij} \qquad \forall (i, j) \in E \\ & x_i \in \{0, 1\} \qquad \forall i \in [n] \\ & e_{ij} \in \{0, 1\} \qquad \forall (i, j) \in E \end{aligned}$$

Relaxing this IP to an LP means changing the last two constraints to

$$\begin{aligned} x_i &\in [0, 1] \\ e_{ij} &\in [0, 1] \end{aligned}$$

Let's focus for the rounding now. Let $\{x_i^*\}, \{e_{ij}^*\}$ be the values of the optimal solutions to the relaxed LP, and let $\{y_i\}, \{z_{ij}\}$ be the corresponding rounded values.

Here's a first rounding attempt. Let's consider the naive rounding where we set z_{ij} to 1 if $e_{ij}^* \geq 1/2$ and $z_{ij} = 0$ otherwise. How does this rounding affect the rounding of x_i^*. We only need to consider the x_i^* affected by the rounding of e_{ij}^* to 1, in order to satisfy the constraint $x_i + x_j \geq e_{ij}$.

In each constraint, we must have $x_i^* + x_j^* \geq 1/2$, so either $x_i^* \geq 1/4$ or $x_j^* \geq 1/4$. Therefore, for $i \in [n]$, we set y_i to 1 if $x_i^* \geq 1/4$, and 0 otherwise.

This rounding scheme satisfies all the IP constraints. Next we bound the rounded variables given our scheme:

$$\begin{aligned} 1 - z_{ij} &\leq 2(1 - e^*_{ij}) & \forall (i,j) \in E \\ y_i &\leq 4x^*_i & \forall i \in [n] \end{aligned}$$

Summing over all variables, the value of the rounded solution is

$$\begin{aligned} \sum_{i \in S} w(i) y_i + \sum_{\substack{(i,j)\in E \\ i,j \notin S}} c(i,j)(1 - z_{ij}) &\leq 4 \sum_{i \in S} w(i) x^*_i + 2 \sum_{\substack{(i,j)\in E \\ i,j \notin S}} c(i,j)(1 - e^*_{ij}) \\ &\leq 4LP_{OPT} \\ &\leq 4IP_{OPT} \end{aligned}$$

Using this rounding scheme, we get a 4-approximation algorithm. But we can do better! In particular, we can just be "vague" about the rounding threshold and decide later what value works best. Formally, let $\alpha \in [0,1]$ be the threshold we use for rounding $\{e^*_{ij}\}$. If $e^*_{ij} \geq \alpha$, then $z_{ij} = 1$, otherwise $z_{ij} = 0$. Using the reasoning above, we set y_i to 1 if $x^*_i \geq \alpha/2$, and 0 otherwise. next we bound these variables given our rounding scheme

$$\begin{aligned} (1 - z_{ij}) &\leq (1-\alpha)^{-1}(1 - e^*_{ij}) \\ y_i &\leq \frac{2}{\alpha} x^*_i \end{aligned}$$

Summing up over all the vertices and edges again we get

$$\sum_{i \in S} w(i) y_i + \sum_{\substack{(i,j)\in E \\ i,j \notin S}} c(i,j)(1 - z_{ij}) \leq \frac{2}{\alpha} \sum_{i \in S} w(i) x^*_i + (1-\alpha)^{-1} \sum_{\substack{(i,j)\in E \\ i,j \notin S}} c(i,j)(1 - e^*_{ij}) \tag{1}$$

This is optimized when $\frac{1}{\alpha} = (1-\alpha)^{-1}$. Solving for α, we get $\alpha = 2/3$. Plugging this value into (1) gives us

$$3 \sum_{i \in S} w(i) x^*_i + 3 \sum_{\substack{(i,j)\in E \\ i,j \notin S}} c(i,j)(1 - e^*_{ij}) \leq 3LP_{OPT} \leq 3IP_{OPT} \tag{2}$$

Therefore this rounding gives a 3-approximation, and since we solved for the best $\alpha \in [0,1]$, this is the best approximation ratio we can get.