CSC 373 - Algorithm Design, Analysis, and Complexity **Summer 2016**

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Network Flows: Bipartite Matching

We conclude our discussion of network flows with an application to bipartite matching. We need the following definitions:

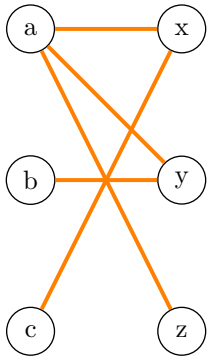
A graph $G(V, E)$ is a **bipartite** graph if V can be partitioned into two sets A and B, such that $A \cup B = V$, and for all $e = (a, b) \in E$, $a \in A, b \in B$.

The pair (A, B) is called a **bi-partition**. We'll use $G(A \cup B, E)$ to refer to a bipartite graph with the bi-partition (A, B).

A **matching** in a graph is a set of edges $M \subseteq E$ such that for every pair of edges $e_1, e_2 \in M$, e_1 and e_2 do not share a common end point.

A matching is **maximal** if it cannot be extended to a larger matching, and **maximum** if it has the most edges out of any matching M' of G.

Example: The graph below is a bipartite graph with bi-partition $A = \{a, b, c\}$ and $B = \{x, y, z\}$. The matching $M = \{(a, x), (b, y)\}$ is maximal, whereas $M' = \{(a, z), (b, y), (c, x)\}$ is maximum.



So the problem we are trying to solve is the following:
The Bipartite Matching Problem:
Input: A bipartite graph $G(A \cup B, E)$.
Output: A maximum matching M of G.

But why this problem and how is it related to network flow? This is just to illustrate how Ford-Fulkerson can be applied in different ways. To solve this problem, we will give a **reduction** from the bipartite matching problem to the maximum flow problem. That is, we will (1) somehow change our bipartite matching problem into a max-flow problem, compute the max flow f and (2) use this solution (the flow f) to extract a solution to our original problem, namely finding a maximum matching.

In more precise terms: Given any bipartite graph $G(A \cup B, E)$, we will show how to construct a flow network (G', s, t, c) such that $\max(f) = |M|$, where M is a maximum matching for $G(A \cup B, E)$. We will then use this max flow f on G' to reconstruct the corresponding maximum matching M on G. This happens in two steps, which we will present in two separate algorithms:

Algorithm 1 ContrusctNetwork

Input: A bipartite graph $G(A \cup B, E)$

Output: A flow network (G', s, t, c) such that $val(f)$, the max flow f of G', equals the size of a maximum matching on G.

1: Construct a flow network $(G'(V', E'), s, t, c)$ as follows:
2: $V' = A \cup B \cup \{s, t\}$ ▷ The vertices of G' are the same as the vertices of G, plus a source and a sink
3: For every $a \in A$, add the edge (s, a) to E'
4: For every $b \in B$, add the edge (b, t) to E'
5: For every $(a, b) \in E$, add (a, b) to E'
6: Set the capacity $c(e') = 1$ for every edge $e' \in E'$

Algorithm 2 ExtractMatching

Input: A flow network (G', s, t, c) and a max flow f on G'

Output: A maximum matching M on G

1: **for** every edge $(a, b) \in G$[1] with flow $f(a, b) = 0$ **do**
2: Remove (a, b) from G.
3: **end for**
4: **Return** the set of edges with $f(a, b) = 1$.

What's the complexity of these two algorithms? Well, $ContrusctNetwork$ takes $\mathcal{O}(m + n)$; one iteration through the vertices and edges of G suffices, and $ExtractMatching$ takes $\mathcal{O}(m)$ time to select the edges with nonzero flow. The final algorithm looks like this now:

Algorithm 3 Bipartite Matching

Input: A bipartite graph $G(A \cup B, E)$

Output: A maximum matching M on G

1: $(G', s, t, c) \leftarrow ContrusctNetwork(G)$
2: $f \leftarrow FordFulkerson(G', s, t, c)$
3: $M \leftarrow ExtractMatching(G', f)$
4: **Return** M

and takes $\mathcal{O}(m^2 n)$ time. Why?

Let's prove that all of this actually works. To show that the reduction works, we'll prove the following theorem:

Theorem: Let $G(A \cup B, E)$ be a bipartite graph. Let (G', s, t, c) be a flow networks constructed by $ContrusctNetwork$ on G:

1. The size of the maximum matching M of G equals the value of the maximum flow f on G'.
2. $ExtractMatching$ returns a maximum matching when given the max flow f on G'.

Proof. 1. Let f be the maximum flow on G' and let a be any element of A. Notice that if $f(s, a) = 1$ then there must exist a vertex b such that $f(a, b) = 1$ (by conservation of flow). By construction of G', we know there exists an edge (b, t) with capacity 1, therefore $f(b, t) = 1$. So this augmenting path $s - a - b - t$ matched vertex a to vertex b. Since $c(s, a) = 1$, we know there doesn't exist a vertex b' with $f(a, b') = 1$ otherwise

we violate the conservation of flow property. So a is matched to b only. And using the same argument on b, we conclude that b is matched to a only. Therefore, the set $M = \{(a, b) \in G' | a \in A, b \in B, f(a, b) = 1\}$ must be a matching in G.

2. Now suppose M, the matching returned by $ExtractMatching$ is not maximum on G, and let M' be a maximum matching for G. Therefore $|M'| > |M|$.
Now construct a flow f' as follows: For every $(a, b) \in M'$, we set $f'(a, b) = 1$. By the previous argument, f' is a valid flow in G', and $val(f') = |M'| > |M| = val(f)$. This contradicts the maximality of f on G' ! $\square$