

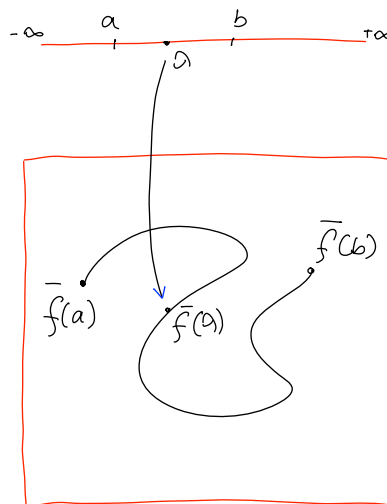
Topic 5:

3D Objects

- General curves & surfaces in 3D
- Normal vectors, surface curves & tangent planes
- Implicit surface representations
- Example surfaces:
surfaces of revolution, bilinear patches, quadrics

Reminder: Curves in 2D

Space of the
curve parameter
(1D)



$$\Delta \in (a, b) \in \mathbb{R}$$

$$\vec{f}(\Delta) \in \mathbb{R}^2$$

where

$$\vec{f}(\Delta) = (x(\Delta), y(\Delta))$$

Space of the
curve (2D)

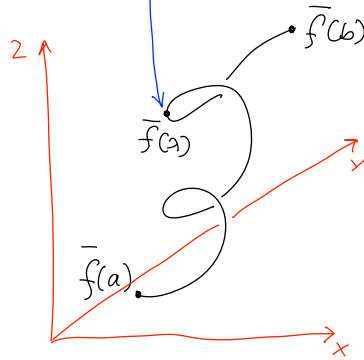
Curves in 3D

Space of the curve parameter (1D)



$$\alpha \in (a, b) \subseteq \mathbb{R}$$

Space of the curve (3D)



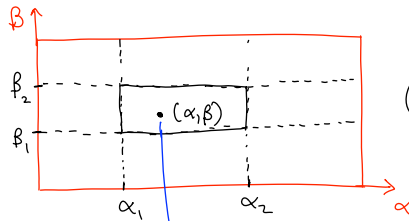
$$\vec{f}(\alpha) \in \mathbb{R}^3$$

where

$$\vec{f}(\alpha) = (x(\alpha), y(\alpha), z(\alpha))$$

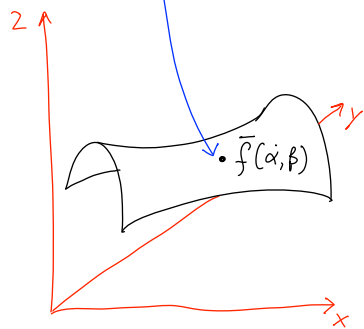
Surfaces in 3D

Space of the surface parameters



$$(\alpha, \beta) \in [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$$

Space of the surface (3D)

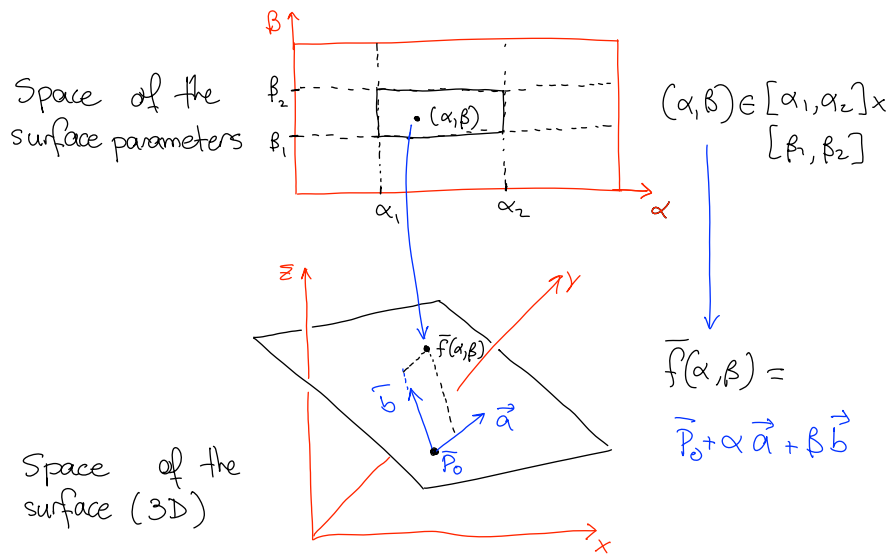


$$\vec{f}(\alpha, \beta) \in \mathbb{R}^3$$

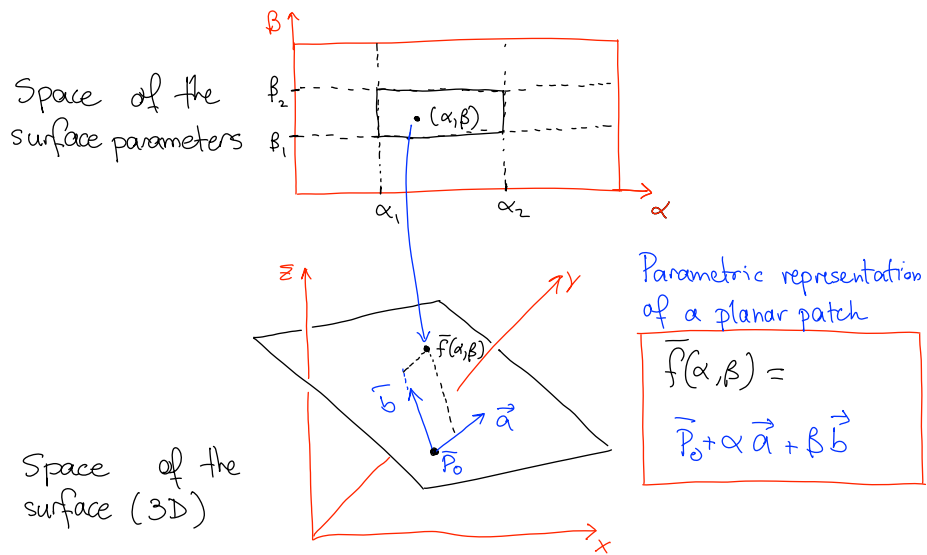
where

$$\vec{f}(\alpha, \beta) = (x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta))$$

Surface Example: Planes in 3D



Surface Example: Planes in 3D



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Normal Vector of a Plane

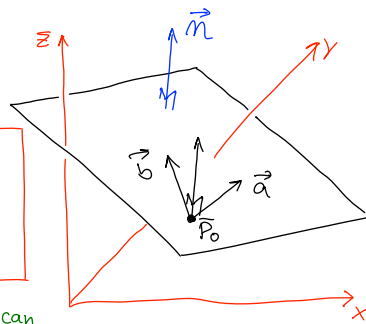
Definition

The normal vector of a plane is any vector that is perpendicular to it (i.e. normal \vec{n} is such that $\vec{n} \cdot \vec{a} = \vec{n} \cdot \vec{b} = 0$)

Normal vector of a plane

$$\vec{n} = \vec{a} \times \vec{b}$$

reminder: cross product can be implemented as a matrix-vector multiplication

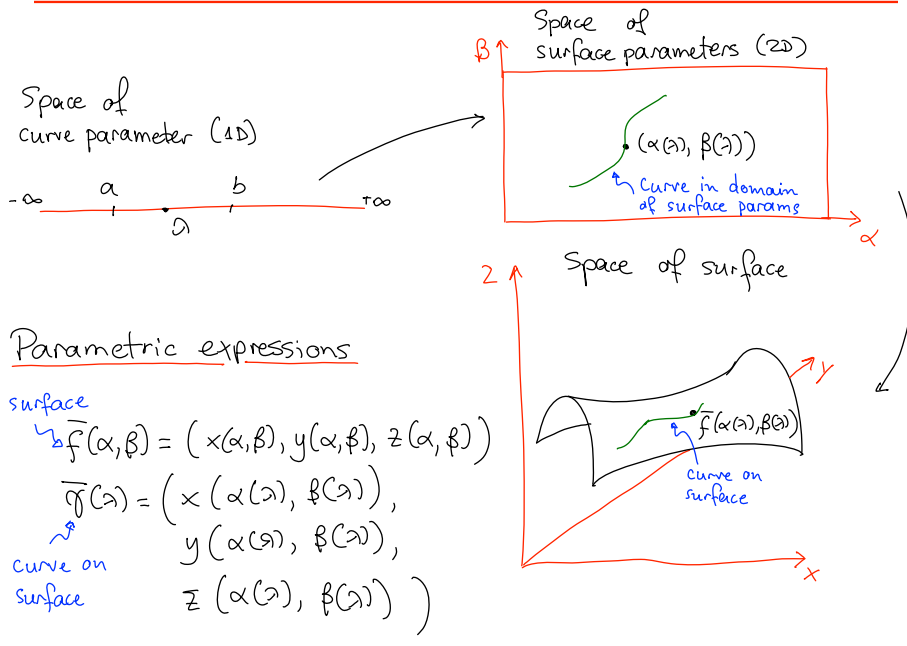


Parametric representation of a planar patch

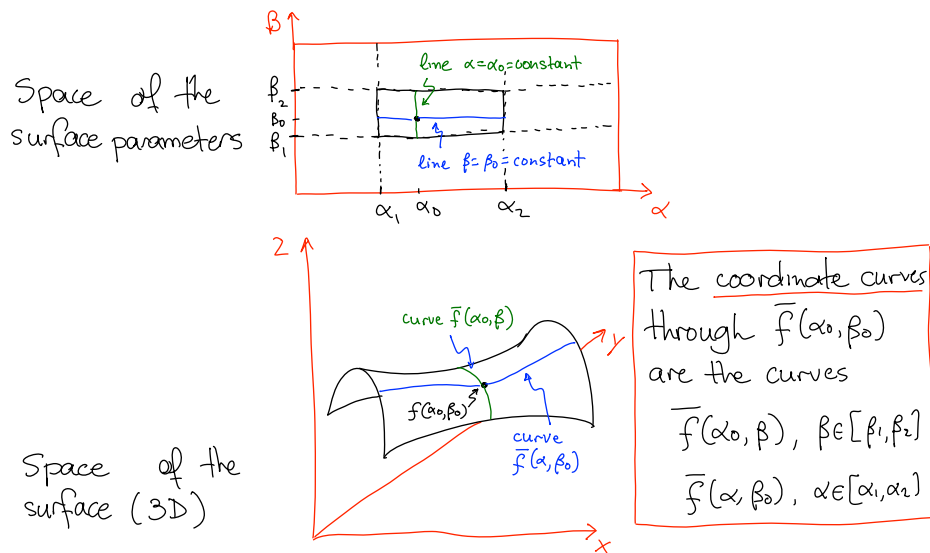
$$\vec{r}(\alpha, \beta) =$$

$$\vec{P}_0 + \alpha \vec{a} + \beta \vec{b}$$

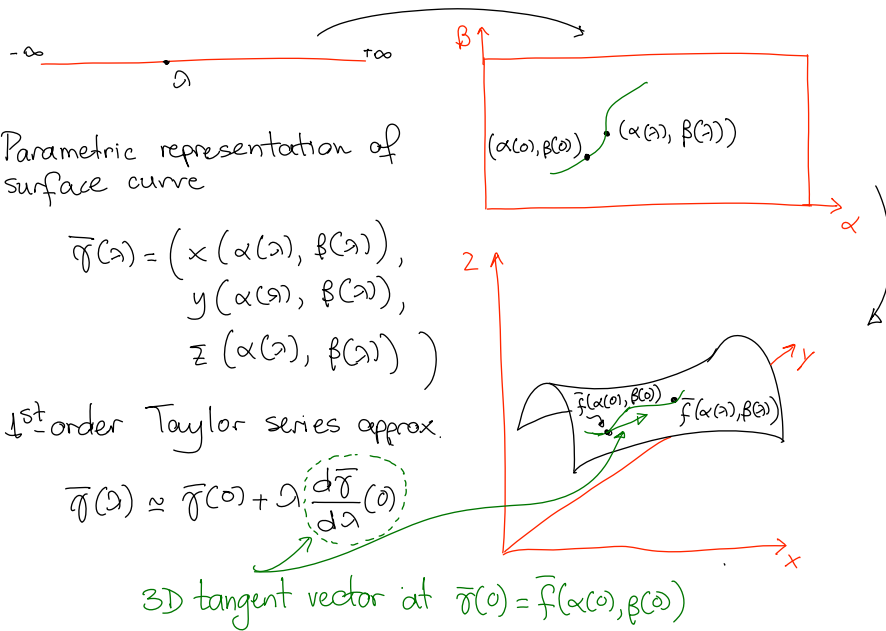
Surface Curves: Parametric Representation



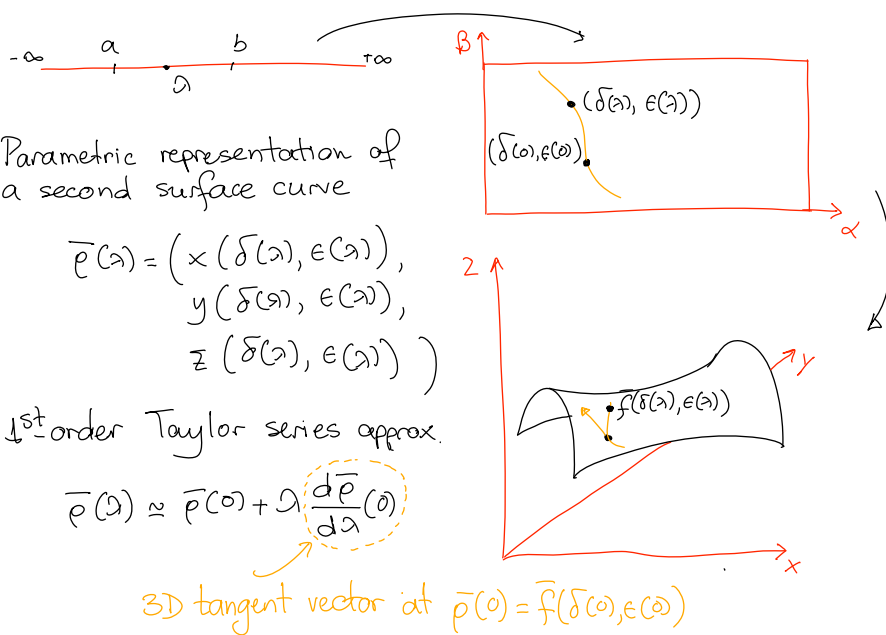
The Coordinate Curves of a Surface



The Tangent Vector of a Surface Curve



The Tangent Vector of a Surface Curve



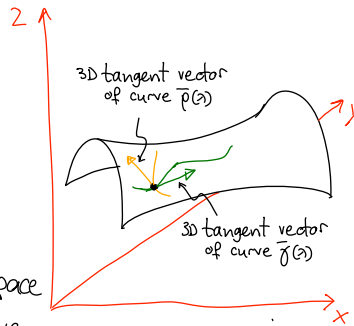
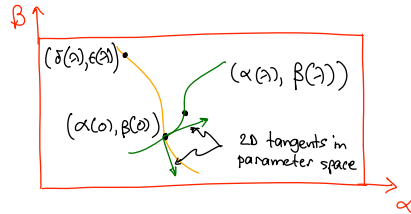
The Set of All Curve Tangents at a Point

Question: What is the relation between

- the 3D tangent of a surface curve
- the corresponding 2D curve in the parameter space
- the 3D surface on which the curve lies?

Ans (intuitive)

- 3D tangent depends on 2D tangent in parameter space
- The set of all possible curve tangents spans a plane



The Set of All Curve Tangents at a Point

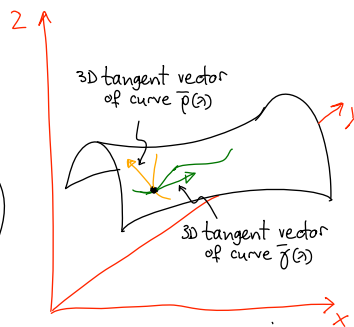
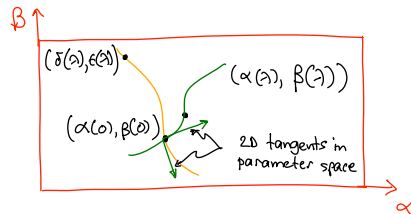
Surface: $\vec{f}(\alpha, \beta) = (x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta))$

Curve: $\vec{r}(s) = (x(\alpha(s), \beta(s)), y(\alpha(s), \beta(s)), z(\alpha(s), \beta(s)))$

3D tangent: $\frac{d\vec{r}}{ds} = \vec{r}'(\alpha(s), \beta(s))$

applying the chain rule, we get

$$\begin{aligned} \frac{d\vec{r}}{ds} &= \left(\frac{\partial x}{\partial \alpha} \frac{d\alpha}{ds} + \frac{\partial x}{\partial \beta} \frac{d\beta}{ds}, \right. \\ &\quad \left. \frac{\partial y}{\partial \alpha} \frac{d\alpha}{ds} + \frac{\partial y}{\partial \beta} \frac{d\beta}{ds}, \right. \\ &\quad \left. \frac{\partial z}{\partial \alpha} \frac{d\alpha}{ds} + \frac{\partial z}{\partial \beta} \frac{d\beta}{ds} \right) \\ &= \frac{\partial \vec{f}}{\partial \alpha} \frac{d\alpha}{ds} + \frac{\partial \vec{f}}{\partial \beta} \frac{d\beta}{ds} \end{aligned}$$

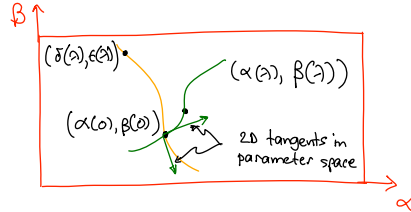


The Set of All Curve Tangents at a Point

Surface: $\vec{f}(\alpha, \beta) = (x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta))$

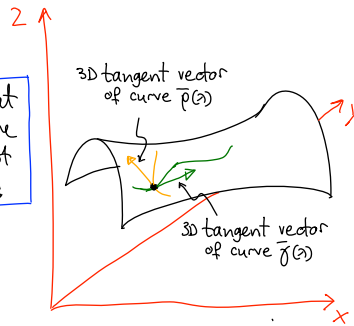
Curve: $\vec{\gamma}(\lambda) = (x(\alpha(\lambda), \beta(\lambda)), y(\alpha(\lambda), \beta(\lambda)), z(\alpha(\lambda), \beta(\lambda)))$

3D tangent: $\frac{d\vec{\gamma}}{d\lambda} = \vec{\gamma}'(\alpha(\lambda), \beta(\lambda))$



3D vectors that depend only on the surface, NOT the curve itself!

scalars that depend on the curve but not the surface!



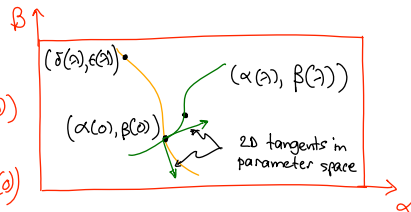
$$\frac{d\vec{\gamma}}{d\lambda} = \frac{\partial \vec{f}}{\partial \alpha} \frac{d\alpha}{d\lambda} + \frac{\partial \vec{f}}{\partial \beta} \frac{d\beta}{d\lambda}$$

The Tangent Plane of the Surface at a Point

The tangent of every surface curve through a point $\vec{f}(\alpha(0), \beta(0))$ is

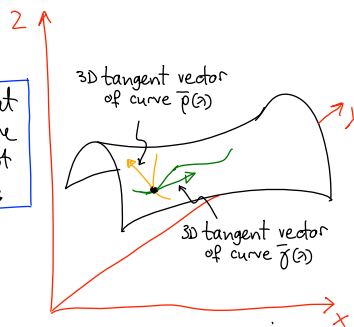
a linear combination of the two 3D vectors $\frac{\partial \vec{f}}{\partial \alpha}(\alpha(0), \beta(0))$ and $\frac{\partial \vec{f}}{\partial \beta}(\alpha(0), \beta(0))$

\Rightarrow they span a plane called the tangent plane at $\vec{f}(\alpha(0), \beta(0))$



3D vectors that depend only on the surface, NOT the curve itself!

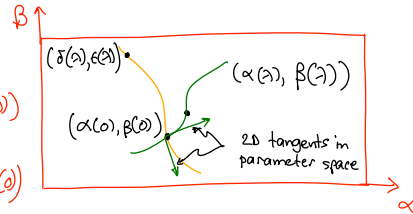
scalars that depend on the curve but not the surface!



$$\frac{d\vec{\gamma}}{d\lambda} = \frac{\partial \vec{f}}{\partial \alpha} \frac{d\alpha}{d\lambda} + \frac{\partial \vec{f}}{\partial \beta} \frac{d\beta}{d\lambda}$$

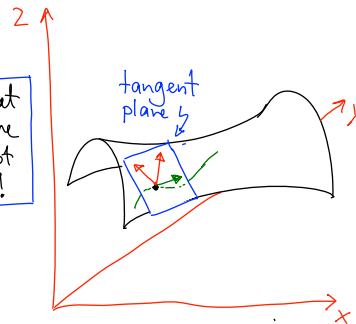
The Tangent Plane of the Surface at a Point

The tangent of every surface curve through a point $\bar{f}(\alpha(\gamma), \beta(\gamma))$ is a linear combination of the two 3D vectors $\frac{\partial \bar{f}}{\partial \alpha}(\alpha(\gamma), \beta(\gamma))$ and $\frac{\partial \bar{f}}{\partial \beta}(\alpha(\gamma), \beta(\gamma))$
 \Rightarrow they span a plane called the **tangent plane at $\bar{f}(\alpha(\gamma), \beta(\gamma))$**



3D vectors that depend only on the surface, NOT the curve itself!

scalars that depend on the curve but not the surface!

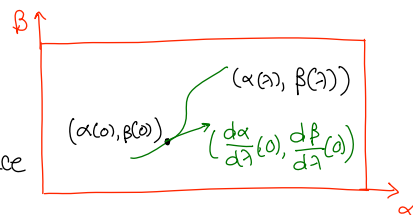


$$\frac{d\bar{f}}{d\gamma} = \frac{\partial \bar{f}}{\partial \alpha} \frac{d\alpha}{d\gamma} + \frac{\partial \bar{f}}{\partial \beta} \frac{d\beta}{d\gamma}$$

The Tangent of a Surface Curve: Geometry

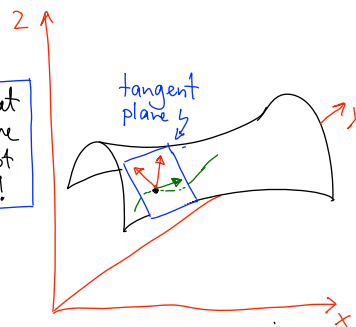
Question: What do the scalars $\frac{d\alpha}{d\gamma}$, $\frac{d\beta}{d\gamma}$ represent geometrically?

Ans: They define the 2D tangent in parameter space



3D vectors that depend only on the surface, NOT the curve itself!

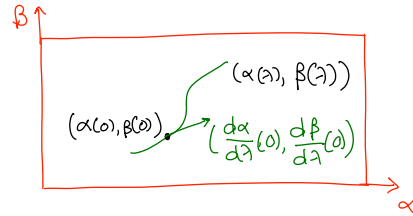
scalars that depend on the curve but not the surface!



$$\frac{d\bar{f}}{d\gamma} = \frac{\partial \bar{f}}{\partial \alpha} \frac{d\alpha}{d\gamma} + \frac{\partial \bar{f}}{\partial \beta} \frac{d\beta}{d\gamma}$$

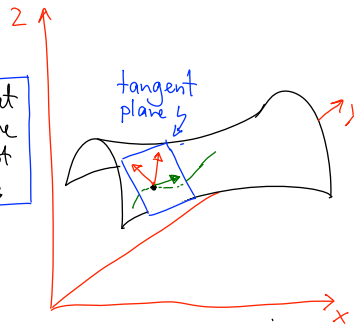
The Tangent of a Surface Curve: Geometry

The tangent at parameter point $(\alpha(\tau), \beta(\tau))$ determines the linear combination of vectors $\frac{\partial \bar{f}}{\partial \alpha}$ and $\frac{\partial \bar{f}}{\partial \beta}$



3D vectors that depend only on the surface, NOT the curve itself!

scalars that depend on the curve but not the surface!



$$\frac{d\bar{\gamma}}{d\tau} = \frac{\partial \bar{f}}{\partial \alpha} \frac{d\alpha}{d\tau} + \frac{\partial \bar{f}}{\partial \beta} \frac{d\beta}{d\tau}$$

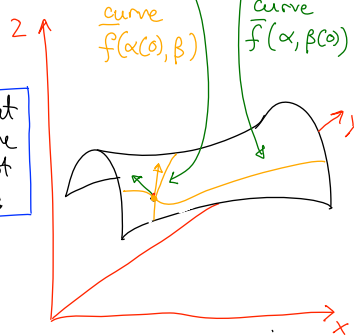
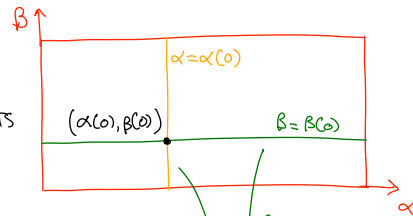
The Tangent of a Surface Curve: Geometry

Question: What do the vectors $\frac{\partial \bar{f}}{\partial \alpha}$, $\frac{\partial \bar{f}}{\partial \beta}$ represent geometrically?

Ans: They are the 3D tangents of the coordinate curves at $(\alpha(\tau), \beta(\tau))$. (Exercise: prove this!)

3D vectors that depend only on the surface, NOT the curve itself!

scalars that depend on the curve but not the surface!



$$\frac{d\bar{\gamma}}{d\tau} = \frac{\partial \bar{f}}{\partial \alpha} \frac{d\alpha}{d\tau} + \frac{\partial \bar{f}}{\partial \beta} \frac{d\beta}{d\tau}$$

The Surface Normal at a Point

The tangent plane at $\bar{f}(\alpha, \beta)$ is the plane spanned by vectors $\frac{\partial \bar{f}}{\partial \alpha}(\alpha, \beta)$ and $\frac{\partial \bar{f}}{\partial \beta}(\alpha, \beta)$

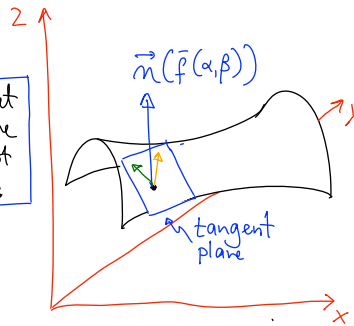
The surface normal at $\bar{f}(\alpha, \beta)$ is the normal to the tangent plane:

$$\vec{n}(\bar{f}(\alpha, \beta)) = \frac{\partial \bar{f}}{\partial \alpha}(\alpha, \beta) \times \frac{\partial \bar{f}}{\partial \beta}(\alpha, \beta)$$

3D vectors that depend only on the surface, NOT the curve itself!

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$$\frac{d\bar{f}}{d\eta} = \frac{\partial \bar{f}}{\partial \alpha} \frac{d\alpha}{d\eta} + \frac{\partial \bar{f}}{\partial \beta} \frac{d\beta}{d\eta}$$



Topic 5:

3D Objects

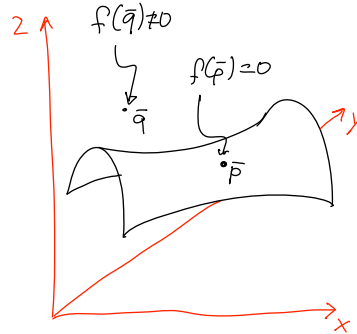
- General curves & surfaces in 3D
- Normal vectors, surface curves & tangent planes
- **Implicit surface representations**
- Example surfaces:
surfaces of revolution, bilinear patches, quadrics

Representing Surfaces by an Implicit Function

- Representation consists of a scalar function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ (called the Implicit Function)
- Surface defined as the set

$$\alpha_0 = \{ \bar{p} \in \mathbb{R}^3 \mid f(\bar{p}) = 0 \}$$

- Intuitively, f can be thought of as measuring a "distance" to the surface
- Typically, f does NOT measure Euclidean distance to the surface

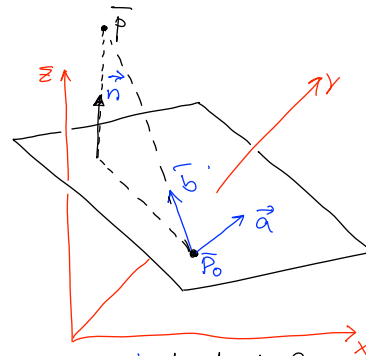


Example: The Implicit Function of a Plane

- Representation consists of a scalar function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$
- Surface defined as the set

$$\alpha_0 = \{ \bar{p} \in \mathbb{R}^3 \mid f(\bar{p}) = 0 \}$$

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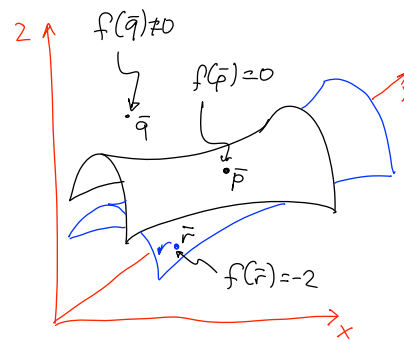
Example: Implicit function for a plane through \bar{p}_0 with normal \vec{n} :

$$f(\bar{p}) = (\bar{p} - \bar{p}_0) \cdot \vec{n}$$

The Level Sets of an Implicit Function

- Representation consists of a scalar function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$
- Surface defined as the set $\alpha_0 = \{\bar{p} \in \mathbb{R}^3 \mid f(\bar{p}) = 0\}$
- Intuitively, f can be thought of as measuring a "distance" to the surface

Given a value $c \in \mathbb{R}$, the set $\alpha_c = \{\bar{p} \in \mathbb{R}^3 \mid f(\bar{p}) = c\}$ also defines a surface called the c -level set of f



Example: Point \bar{r} belongs to the -2 -level set of f .

Surface Normals from the Implicit Function

If \bar{p} is a surface point, the normal at \bar{p} is given by

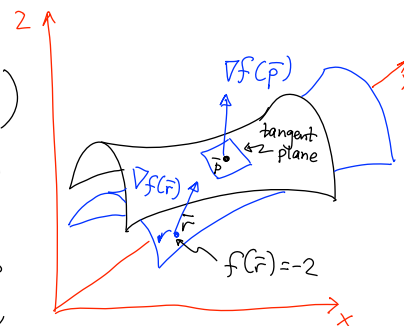
$$\vec{n}(\bar{p}) = \nabla f(\bar{p})$$

where

$$\nabla f(\bar{p}) = \left(\frac{\partial f}{\partial x}(\bar{p}), \frac{\partial f}{\partial y}(\bar{p}), \frac{\partial f}{\partial z}(\bar{p}) \right)$$

(a.k.a. the gradient of f)

Note: the above definition works for any level set: if $\bar{r} \in \alpha_c$, the normal of the c -level-set at point \bar{r} is given by $\nabla f(\bar{r})$



Surface Normals from the Implicit Function

Proof: Let $\bar{q}(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$ be a curve on the surface α_c with $\bar{q}(0) = \bar{r}$.

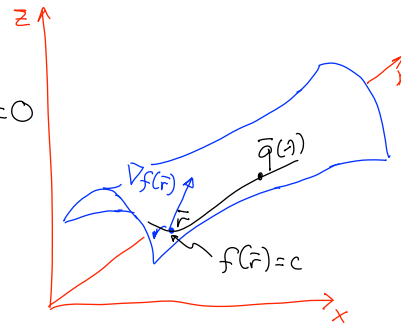
$$\Leftrightarrow \forall \lambda \quad f(\bar{q}(\lambda)) = c$$

$$\Leftrightarrow \frac{df}{d\lambda}(\bar{q}(0)) = 0$$

$$\Leftrightarrow \frac{\partial f}{\partial x} \frac{dx}{d\lambda} + \frac{\partial f}{\partial y} \frac{dy}{d\lambda} + \frac{\partial f}{\partial z} \frac{dz}{d\lambda} = 0$$

$$\Leftrightarrow \nabla f(\bar{q}(0)) \cdot \frac{d\bar{q}}{d\lambda}(0) = 0$$

Note: the above definition works for any level set: if $\bar{r} \in \alpha_c$, the normal of the c -level-set at point \bar{r} is given by $\nabla f(\bar{r})$



Surface Normals from the Implicit Function

Proof: Let $\bar{q}(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$ be a curve on the surface α_c with $\bar{q}(0) = \bar{r}$.

$$\Leftrightarrow \forall \lambda \quad f(\bar{q}(\lambda)) = c$$

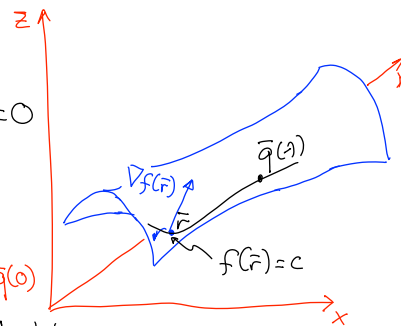
$$\Leftrightarrow \frac{df}{d\lambda}(\bar{q}(0)) = 0$$

$$\Leftrightarrow \frac{\partial f}{\partial x} \frac{dx}{d\lambda} + \frac{\partial f}{\partial y} \frac{dy}{d\lambda} + \frac{\partial f}{\partial z} \frac{dz}{d\lambda} = 0$$

$$\Leftrightarrow \nabla f(\bar{q}(0)) \cdot \frac{d\bar{q}}{d\lambda}(0) = 0$$

gradient at \bar{r} 3D tangent at $\bar{q}(0)$

since the above orthogonality holds for any curve in α_c through \bar{r} , the gradient must be perpendicular to the tangent plane (QED)

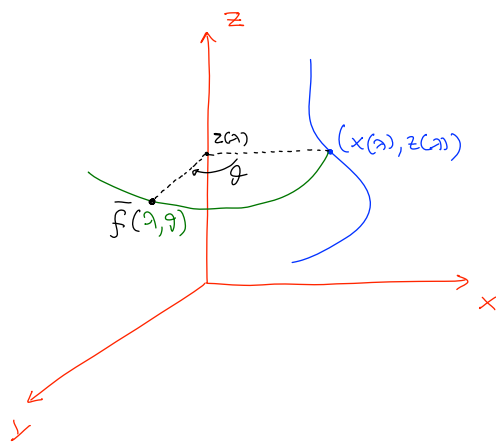


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Surfaces of Revolution: Basic Construction



Conceptual steps:

1. Define a 2D curve on the xz -plane:
2. Rotate it about the z -axis

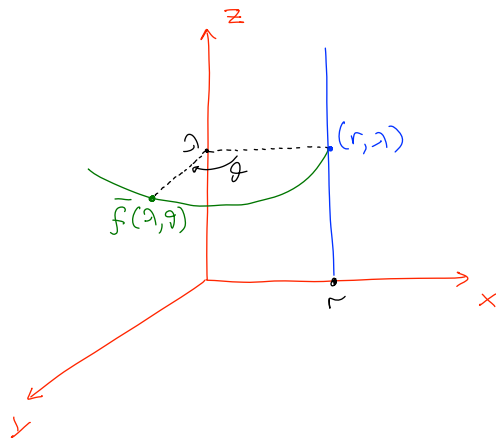
Another equivalent view:

- Point $(x(z), y(z))$ will trace a circle in xy -plane
- The circle's radius is equal to $x(z)$
- θ ranges in $[0, 2\pi)$

Parametric representation

$$\vec{f}(z, \theta) = (x(z)\cos\theta, x(z)\sin\theta, z(z))$$

Example: The Cylinder



Question: how do we express the cylinder of radius r ?

Ans:

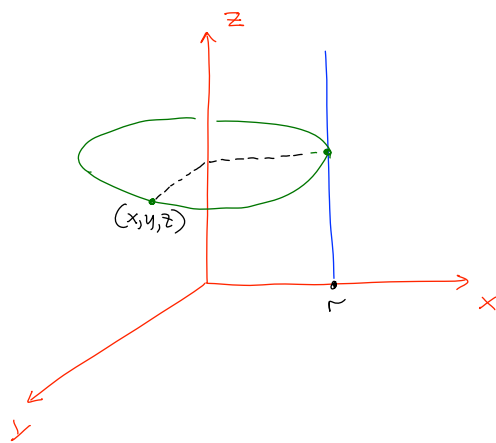
$$(x(\lambda), z(\lambda)) = (r, \lambda)$$

So

$$\vec{f}(\lambda, \vartheta) = (r \cos \vartheta, r \sin \vartheta, \lambda)$$

Parametric representation
 $\vec{f}(\lambda, \vartheta) = (x(\lambda) \cos \vartheta, x(\lambda) \sin \vartheta, z(\lambda))$

Example: Implicit Function of the Cylinder



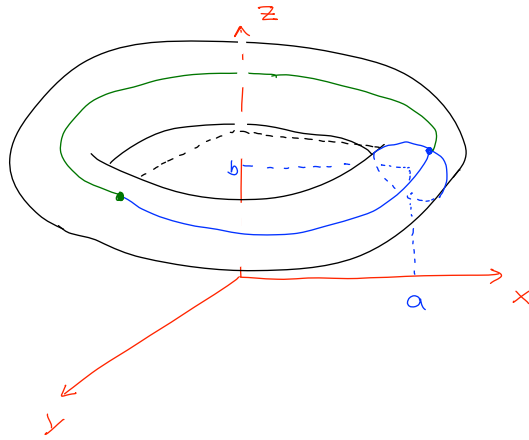
Question: how do we express the cylinder of radius r ?

Ans:

The points (x, y, z) on the cylinder have constant distance r from z -axis

Implicit equation
 $f(x, y, z) = x^2 + y^2 - r^2 = 0$

Example: The Torus as a Surface of Revolution



Question: how do we express the torus as a surface of revolution?

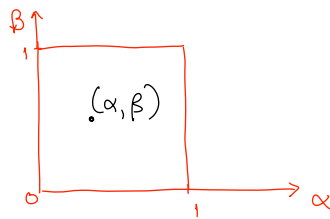
Ans: torus is formed by rotating a circle about the z-axis

$$(x(\vartheta), z(\vartheta)) = (r \cos \vartheta + a, r \sin \vartheta + b)$$

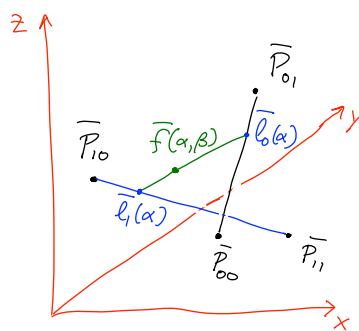
Parametric representation

$$\bar{f}(\vartheta, \vartheta) = (x(\vartheta) \cos \vartheta, x(\vartheta) \sin \vartheta, z(\vartheta))$$

Bilinear Patches: Basic Construction



Space of parameters
 $(\alpha, \beta) \in [0, 1] \times [0, 1]$



Given 4 3D points $\bar{P}_{00}, \bar{P}_{01}, \bar{P}_{10}, \bar{P}_{11}$, the pair (α, β) uniquely determines $\bar{f}(\alpha, \beta)$:

$$\bar{l}_0(\alpha) = (1-\alpha)\bar{P}_{00} + \alpha\bar{P}_{01}$$

$$\bar{l}_1(\alpha) = (1-\alpha)\bar{P}_{10} + \alpha\bar{P}_{11}$$

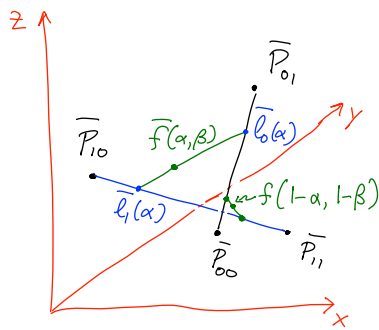
$$\bar{f}(\alpha, \beta) = (1-\beta)\bar{l}_0(\alpha) + \beta\bar{l}_1(\alpha)$$

Bilinear Patches: Basic Construction

Question: What kind of surface do we get?

Is it a plane?

Ans: It is a plane only if $\bar{P}_{00}, \bar{P}_{01}, \bar{P}_{10}, \bar{P}_{11}$ are coplanar!



Given 4 3D points $\bar{P}_{00}, \bar{P}_{01}, \bar{P}_{10}, \bar{P}_{11}$, the pair (α, β) uniquely determines $\bar{f}(\alpha, \beta)$:

$$\bar{l}_0(\alpha) = (1-\alpha)\bar{P}_{00} + \alpha\bar{P}_{01}$$

$$\bar{l}_1(\alpha) = (1-\alpha)\bar{P}_{10} + \alpha\bar{P}_{11}$$

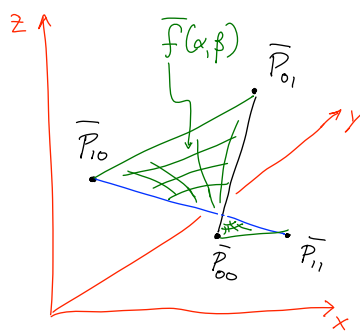
$$\bar{f}(\alpha, \beta) = (1-\beta)\bar{l}_0(\alpha) + \beta\bar{l}_1(\alpha)$$

Bilinear Patches: Basic Construction

Question: What kind of surface do we get?

Is it a plane?

Ans: It is a plane only if $\bar{P}_{00}, \bar{P}_{01}, \bar{P}_{10}, \bar{P}_{11}$ are coplanar!



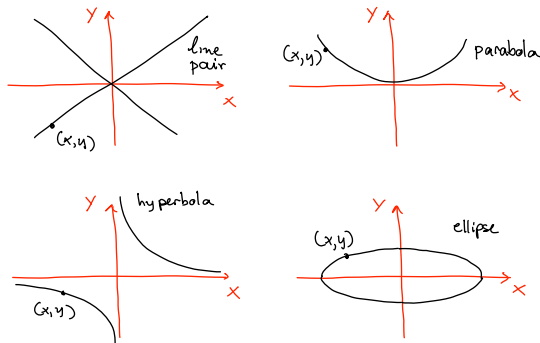
Given 4 3D points $\bar{P}_{00}, \bar{P}_{01}, \bar{P}_{10}, \bar{P}_{11}$, the pair (α, β) uniquely determines $\bar{f}(\alpha, \beta)$:

$$\bar{l}_0(\alpha) = (1-\alpha)\bar{P}_{00} + \alpha\bar{P}_{01}$$

$$\bar{l}_1(\alpha) = (1-\alpha)\bar{P}_{10} + \alpha\bar{P}_{11}$$

$$\bar{f}(\alpha, \beta) = (1-\beta)\bar{l}_0(\alpha) + \beta\bar{l}_1(\alpha)$$

Refresher on (2D) Conic Sections



General implicit equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

or

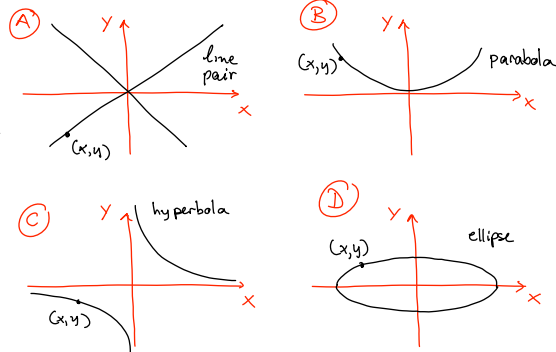
$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} A & C/2 & D/2 \\ C/2 & B & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

shape controlled by params
 A, B, C, D, E, F
 (one of A, B, C must be $\neq 0$)
 same eq but in
 homogeneous/matrix
 notation

Quadric Surfaces: Basic Construction

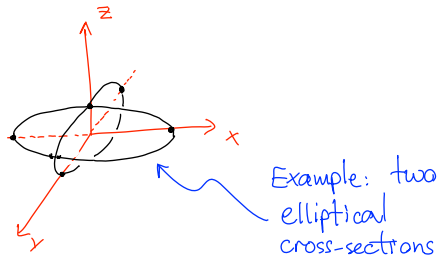
Definition:

Surfaces whose
 planar cross-sections
 are conics



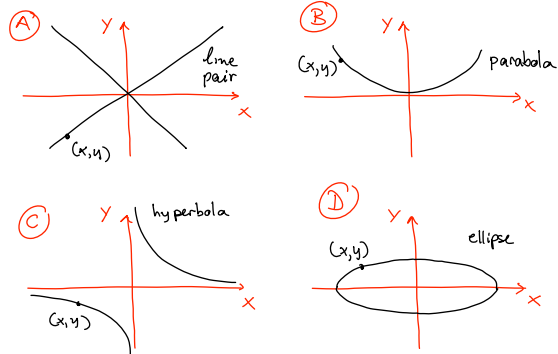
Intuition:

To visualize all possible
 shapes, consider their
 cross-sections with planes
 parallel to xz and
 yz planes



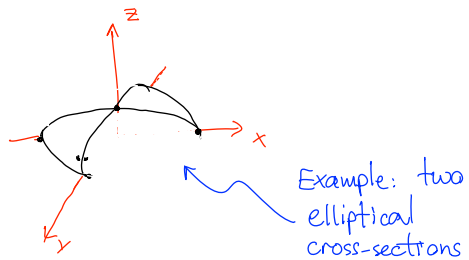
Quadric Surfaces: Basic Construction

Case $\textcircled{A} + \textcircled{D}$:
Ellipsoid



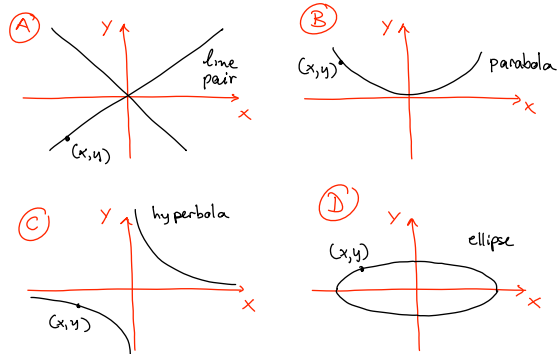
Intuition:

To visualize all possible shapes, consider their cross-sections with planes parallel to xz and yz planes



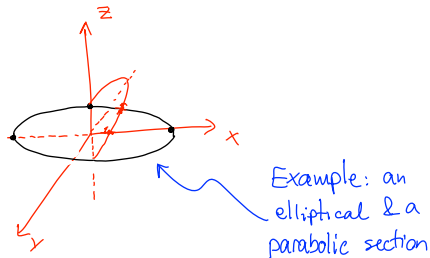
Quadric Surfaces: Basic Construction

Case $\textcircled{B} + \textcircled{D}$:
Elliptical paraboloid



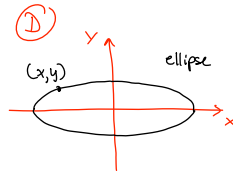
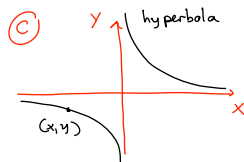
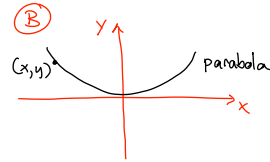
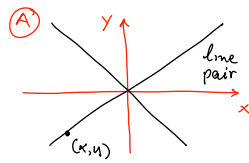
Intuition:

To visualize all possible shapes, consider their cross-sections with planes parallel to xz and yz planes



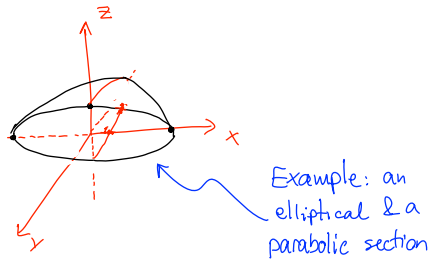
Quadric Surfaces: Basic Construction

Case (B) + (D)
Elliptical
paraboloid



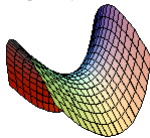
Intuition:

To visualize all possible shapes, consider their cross-sections with planes parallel to xz - and yz -planes

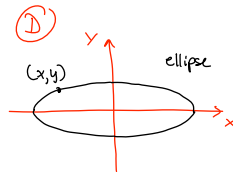
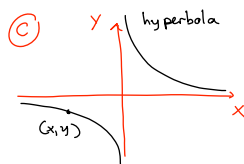
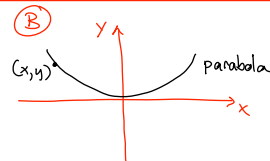
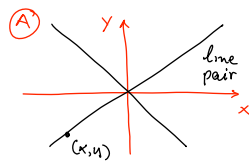
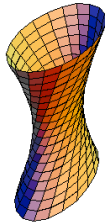


Quadric Surfaces: Basic Construction

(B) + (C)
Hyperbolic
paraboloid



(C) + (D)
Elliptic
hyperboloid



Quadric Surfaces: Implicit Equation

General implicit equation

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} \textcircled{A} & D & E & G \\ D & \textcircled{B} & F & H \\ E & F & \textcircled{C} & \textcircled{I} \\ G & H & \textcircled{I} & \textcircled{J} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

defined up to a scale factor

- 9 total degrees of freedom
- Planar cross-sections produce curves with an implicit eq of a conic section
- Only 5 parameters used (those circled) when cross-sections are "centered" on xy -, yz -, xz -planes

Polygonal Meshes

Definition (polygonal mesh)

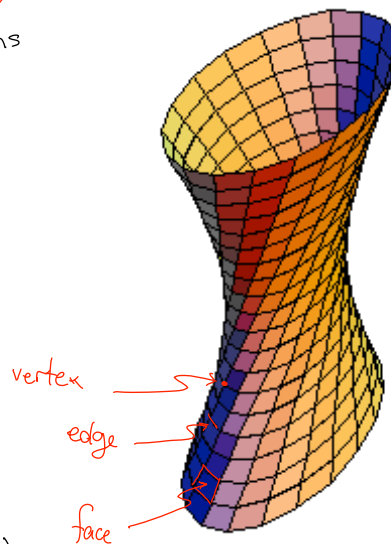
A collection of polygons

- vertices
- edges
- faces

• **Polyhedron**: a closed, connected, polygonal mesh

• **Face**: Planar polygonal patch inside a mesh

• A mesh is **simple** when it has no holes (i.e. topologically equiv to a sphere)



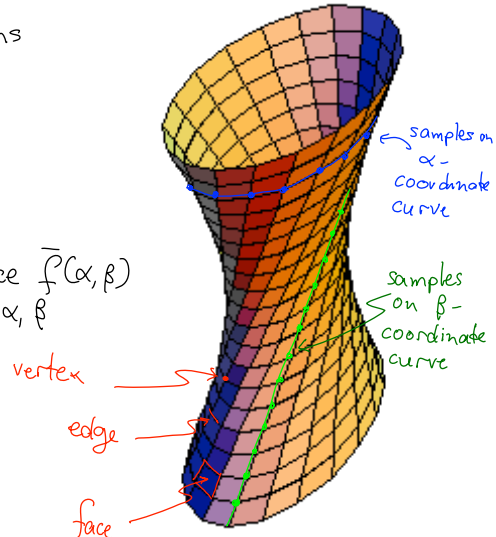
Polygonal Meshes

Definition (polygonal mesh)

A collection of polygons

- vertices
- edges
- faces

Given a parametric surface $\bar{f}(\alpha, \beta)$
we can sample values of α, β
to define a mesh that approximates it



Generating Triangle Meshes

A quadruple of points may not define a planar face



Solution: break up quad into 2 triangles

* See online notes on how to do this properly

* See online notes about what data structure to use to store a mesh.

