Today's Topics

3. Transformations in 2D

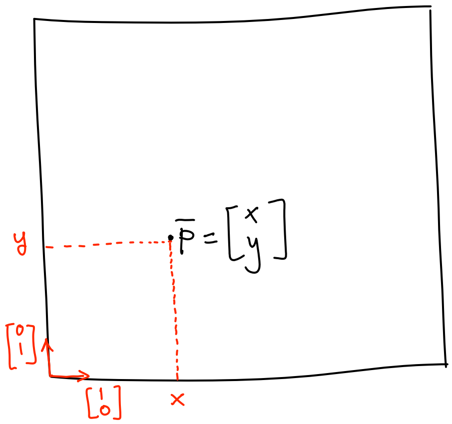
4. Coordinate-free geometry

Topic 3:

2D Transformations

- Homogeneous coordinates
- Homogeneous 2D transformations
- Affine transformations & restrictions

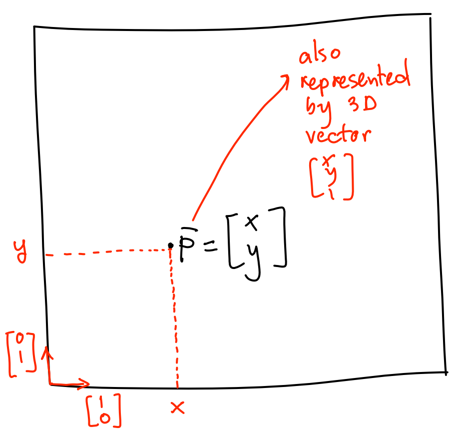
Representing Points by Euclidean 2D Coords



- "Standard" (Euclidean) representation of a point $\bar{p}$:

$$p = x \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{basis vectors}} = \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Euclidean coordinates}}$$

Euclidean Coords ⇒ Homogeneous Coords



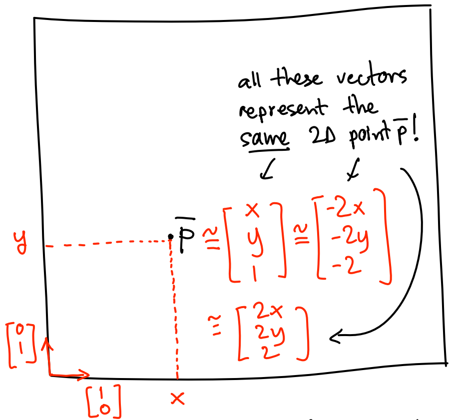
- "Standard" (Euclidean) representation of a point p:

$$\bar{p} = x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- Homogeneous (a.k.a. Projective) representation of $\bar{p}$

$$\boxed{\underset{\text{pixel coordinates}}{\begin{bmatrix} x \\ y \end{bmatrix}} \longrightarrow \underset{\text{homogeneous 2D coordinates}}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}}$$

2D Homogeneous Coordinates: Definition



- For any $\lambda \neq 0$, the numbers $\lambda x, \lambda y, \lambda$ are called the homogeneous coordinates of point $\bar{p}$

Definition:

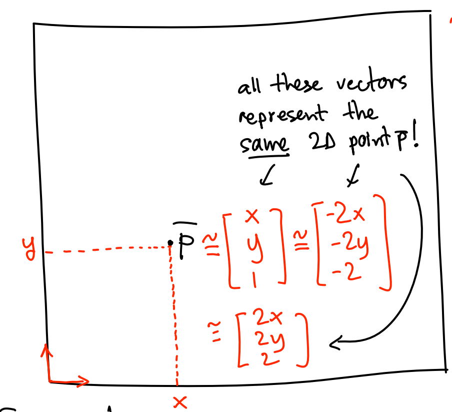
Homogeneous representation of $\bar{p}$

$\boxed{\bar{p} \text{ represented by any 3D vector } \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} \text{ with } \lambda \neq 0}$

- Homogeneous (a.k.a. Projective) representation of $\bar{p}$

pixel coordinates		homogeneous 2D coordinates	
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\longrightarrow$	$\begin{bmatrix} \lambda \cdot x \\ \lambda \cdot y \\ \lambda \cdot 1 \end{bmatrix}$	$\lambda \neq 0$

2D Homogeneous Coordinates: Equality



Examples:

- Is $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \cong \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$? yes (take $\lambda=2$)
- Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 30 \end{bmatrix}$? yes (take $\lambda=30$)
- Is $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cong \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$? no!

Definition (Homogeneous Equality)

Two vectors of homogeneous coords $\bar{v}_1 = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$ are called equal if they represent the same 2D point:

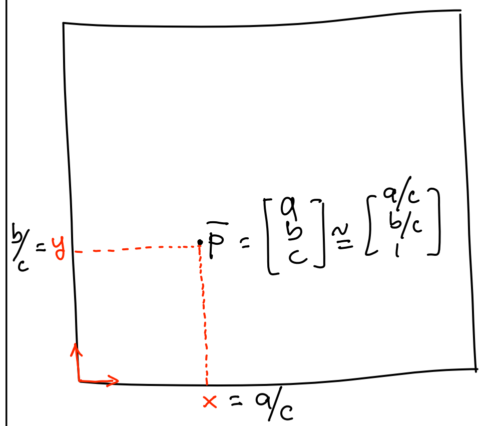
$\bar{v}_1 \cong \bar{v}_2$ denotes homog. equality

$\Longleftrightarrow$

there is a $\lambda \neq 0$ such that

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

Homogeneous Coords $\Rightarrow$ Euclidean Coords



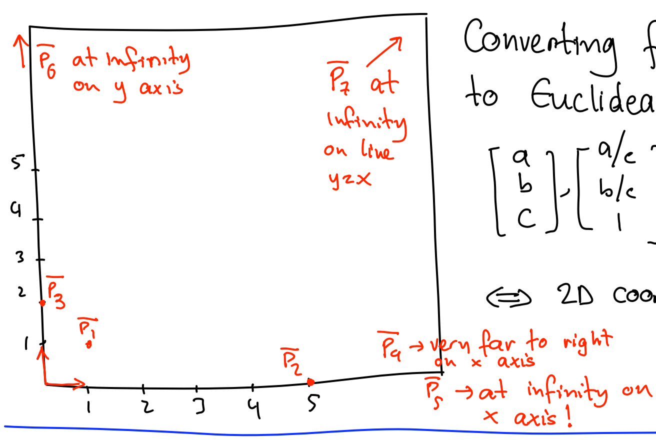
Converting from homogeneous to Euclidean coordinates:

$\begin{bmatrix}a\\b\\c\end{bmatrix}, \begin{bmatrix}a/c\\b/c\\1\end{bmatrix}$ represent the same 2D point

$\Leftrightarrow$ 2D coordinates are $\begin{bmatrix}a/c\\b/c\end{bmatrix}$

$$\boxed{\begin{gathered}\bar{v}_1 \cong \bar{v}_2 \\ \Longleftrightarrow \\ \text{there is a } \lambda \neq 0 \text{ such that} \\ \begin{bmatrix}x\\y\\w\end{bmatrix} = \lambda \begin{bmatrix}x'\\y'\\w'\end{bmatrix}\end{gathered}}$$

Homogeneous Coords $\Rightarrow$ Euclidean Coords



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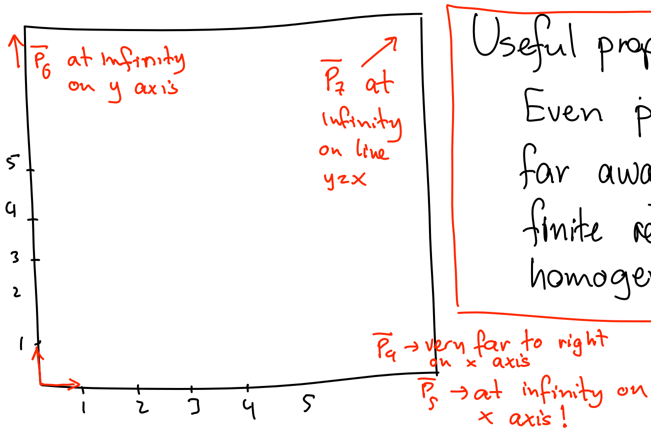
$\Leftrightarrow$ 2D coordinates are $\begin{bmatrix}a/c\\b/c\end{bmatrix}$

Practice exercise: Plot positions of the following points

$\bar{P}_6 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\bar{P}_7 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

$\bar{P}_1 = \begin{bmatrix}2\\2\\2\end{bmatrix}$ $\bar{P}_2 = \begin{bmatrix}10\\0\\2\end{bmatrix}$ $\bar{P}_3 = \begin{bmatrix}0\\8\\4\end{bmatrix}$ $\bar{P}_4 = \begin{bmatrix}1\\0\\0.0001\end{bmatrix}$ $\bar{P}_5 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Points at ∞ in Homogeneous Coordinates



Useful property #1:
Even points infinitely far away have a finite representation in homogeneous coords!

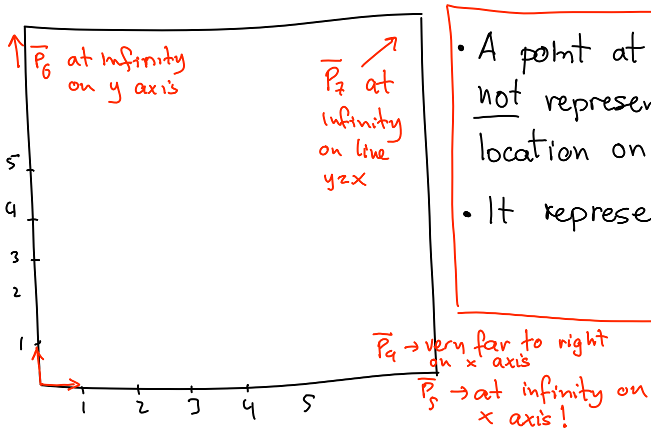
leads to very stable geometric computations

Points at infinity have their last coordinate equal to zero

$$\bar{P}_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \bar{P}_7 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{P}_4 = \begin{bmatrix} 1 \\ 0 \\ 0.0001 \end{bmatrix} \quad \bar{P}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Points at ∞ in Homogeneous Coordinates



- A point at infinity does not represent a physical location on the plane
- It represents a direction

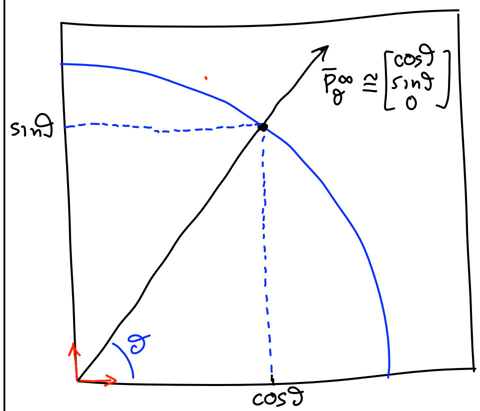
Points at infinity have their last coordinate equal to zero

$$\bar{P}_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \bar{P}_7 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

first two coords represent directions in 2D

$$\bar{P}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Points at ∞ in Homogeneous Coordinates



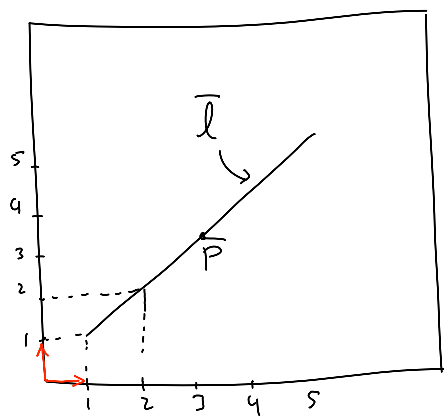
- A point at infinity does <u>not</u> represent a physical location on the plane
- It represents <u>a direction</u>

Points at infinity have their last coordinate equal to zero

$\bar{p}_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\bar{p}_7 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\bar{p}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

first two coords represent directions in 2D

Line Equations in Homogeneous Coordinates



- The equation of a line

$$\underbrace{a}_{\text{line parameters}}x + \underbrace{b}y + \underbrace{c} = 0$$

- In homogeneous coordinates

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or $\bar{\ell}^T \cdot \bar{p} = 0$

vector holding line parameters

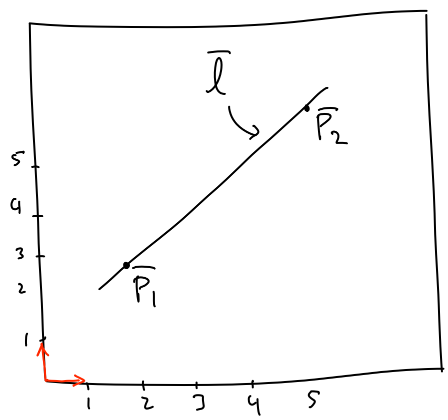
vector holding homogeneous coordinates of a point

Example: line $y=x$ in homogeneous coords:

$$1 \cdot x - 1 \cdot y + 0 \cdot 1 = 0$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}_{\text{line parameters of } \ell} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

The Line Passing Through 2 Points



- Calculating the parameters of a line through two points with homogeneous coordinates $\bar{p}_1, \bar{p}_2$

$$\boxed{\bar{l} = \bar{p}_1 \times \bar{p}_2}$$

↑ cross product of two 3D vectors

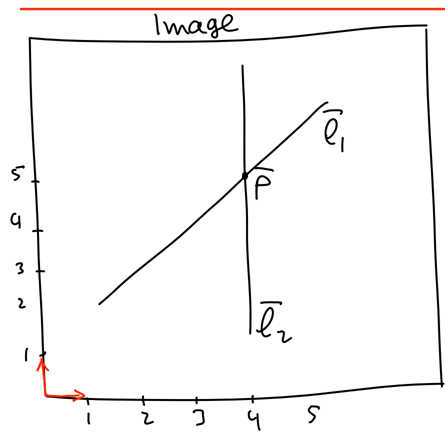
- In homogeneous coordinates

$$[a \ b \ c]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or $\bar{l}^T \cdot \bar{p} = 0$

- $\bar{l}$ must satisfy $\bar{l}^T \cdot \bar{p}_1 = 0$, $\bar{l}^T \bar{p}_2 = 0$
- taken as 3D vectors, $\bar{l}$ is perpendicular to both $\bar{p}_1$ and $\bar{p}_2$ ⇒ it is along the cross product, $\bar{p}_1 \times \bar{p}_2$

The Point of Intersection of Two Lines



- Calculating the homogeneous coordinates of the intersection of two lines $\bar{l}_1, \bar{l}_2$

$$\boxed{\bar{p} = \bar{l}_1 \times \bar{l}_2}$$

↑ cross product of two 3D vectors

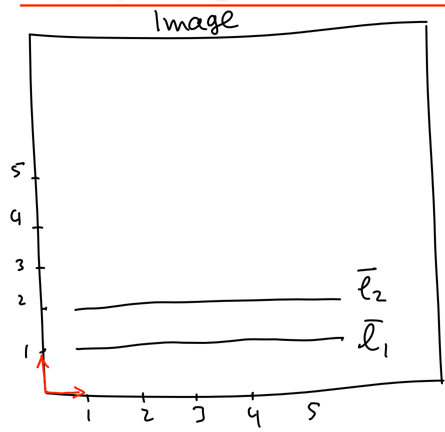
- In homogeneous coordinates

$$[a \ b \ c]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or $\bar{l}^T \cdot \bar{p} = 0$

- $\bar{p}$ must satisfy $\bar{l}_1^T \bar{p} = 0$, $\bar{l}_2^T \bar{p} = 0$
- taken as 3D vectors, $\bar{p}$ is perpendicular to both $\bar{l}_1$ and $\bar{l}_2$ ⇒ it is along the cross product, $\bar{l}_1 \times \bar{l}_2$

Computing the Intersection of Parallel Lines



· Calculating the homogeneous coordinates of the intersection of two lines $\bar{\ell}_1, \bar{\ell}_2$

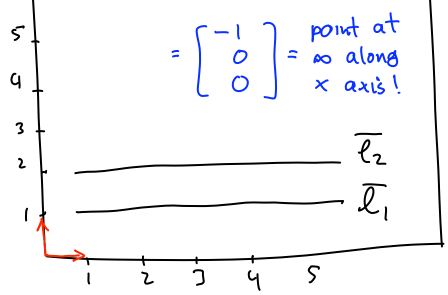
$$\boxed{\bar{p} = \bar{\ell}_1 \times \bar{\ell}_2}$$

↑ cross product of two 3D vectors

This calculation works even when $\bar{\ell}_1, \bar{\ell}_2$ are parallel!

(no floating point exceptions or divide-by-zero errors!)

Computing the Intersection of Parallel Lines



$$\bar{\ell}_1 \times \bar{\ell}_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \text{point at } \infty \text{ along } x \text{ axis!}$$

· Line eq. of $\bar{\ell}_1$ is $y=1$. Also written as $0 \cdot x + 1 \cdot y - 1 = 0$. So $\bar{\ell}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

· Similarly $\bar{\ell}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

· Calculating the homogeneous coordinates of the intersection of two lines $\bar{\ell}_1, \bar{\ell}_2$

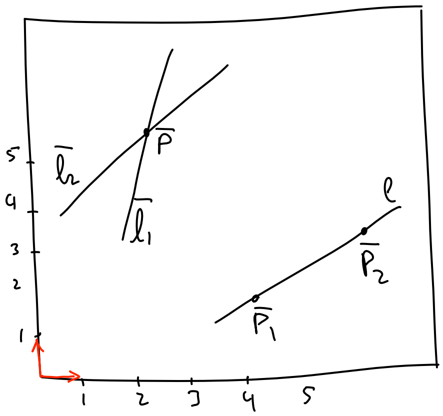
$$\boxed{\bar{p} = \bar{\ell}_1 \times \bar{\ell}_2}$$

↑ cross product of two 3D vectors

Aside (calculating cross products): If $\bar{\ell}_1 = (a, b, c)$ then

$$\bar{\ell}_1 \times \bar{\ell}_2 = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \bar{\ell}_2$$

Lines from Points & Points from Lines



Useful property #2

- Very simple way of computing & intersecting lines
- Numerical stability even when result is at ∞

Line through 2 points

$$\bar{l} = \bar{p}_1 \times \bar{p}_2 = \begin{bmatrix} 0 & -p_1^z & p_1^y \\ p_1^z & 0 & -p_1^x \\ -p_1^y & p_1^x & 0 \end{bmatrix} \begin{bmatrix} p_2^x \\ p_2^y \\ p_2^z \end{bmatrix}$$

Intersection of 2 lines

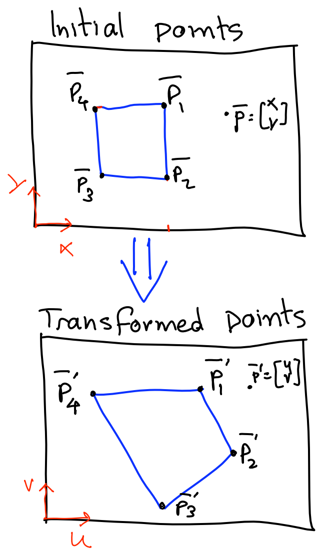
$$\bar{p} = \bar{l}_1 \times \bar{l}_2 = \begin{bmatrix} 0 & -l_1^z & l_1^y \\ l_1^z & 0 & -l_1^x \\ -l_1^y & l_1^x & 0 \end{bmatrix} \begin{bmatrix} l_2^x \\ l_2^y \\ l_2^z \end{bmatrix}$$

Topic 3:

2D Transformations

- Homogeneous coordinates
- Homogeneous 2D transformations
- Affine transformations & restrictions

2D Transformations



Definition:

A function $\bar{f}: \mathbb{R}^2 \to \mathbb{R}^2$

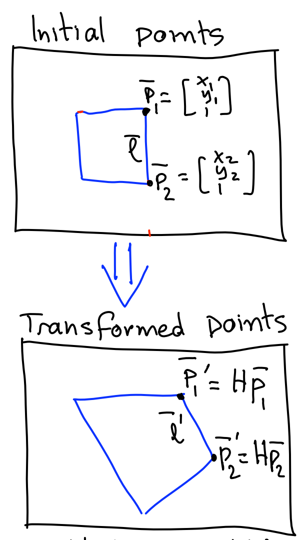
$$\bar{p} \mapsto \bar{f}(p)$$

- Usually f is invertible.
- In this case it can be thought of as a change in coordinates

$$\bar{p} = \begin{bmatrix} x \\ y \end{bmatrix} \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \bar{p}' = \begin{bmatrix} u \\ v \end{bmatrix}$$

- Applications:
  - Compose objects with parts
  - Shape deformation
  - Animation

General Linear 2D Transformations



When H is invertible, it is called a Homography

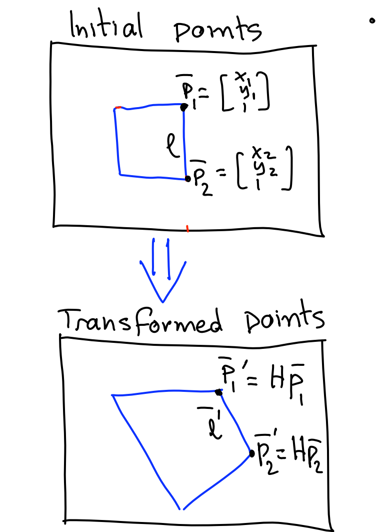
- Definition (Linear 2D Transforms)

  A 2D transform is called linear if every 2D line $\bar{\ell}$ in the original image is transformed into a line $\bar{\ell}'$ in the warped image (i.e. the warp preserves all lines in the original photo)

- Property (w/out proof)

  Every linear warp can be expressed as a 3x3 matrix H that transforms homogeneous image coordinates

General Linear 2D Transformations



When H is invertible, it is called a Homography

- So our focus will be on transformations $f()$ for which

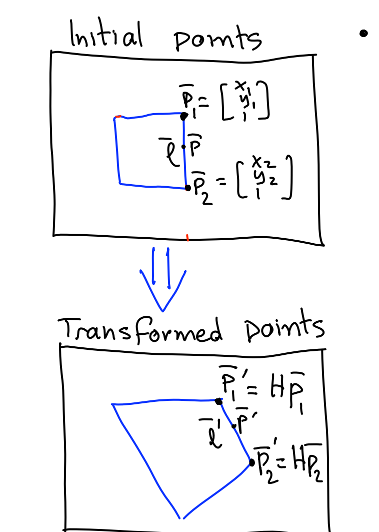
$$\boxed{f(\bar p) = H\bar p}$$

for some homography matrix H

- Property (w/out proof)

Every linear warp can be expressed as a 3x3 matrix H that transforms homogeneous image coordinates

Homographies: Basic properties



So the transformed points satisfy a line equation too, with line coordinates $\bar\ell^T H^{-1}$

- Homographies transform lines to lines

Proof:

. All points on line ℓ satisfy

$$\bar\ell^T \cdot \bar p = 0 \qquad (*)$$

. The homography H will transform $\bar p$ to $H\bar p$. Therefore,

$$\bar p' \cong H\bar p \iff \bar p \cong H^{-1}\bar p'$$

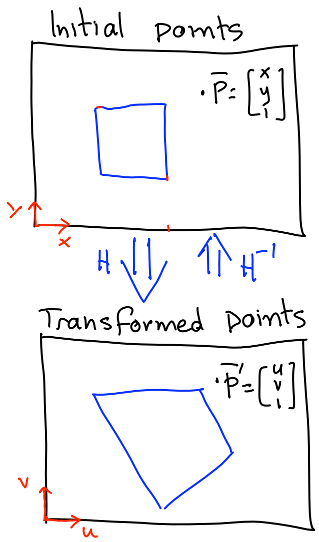
. Combining with (*) we get

$$\bar\ell^T \cdot H^{-1}\bar p' = 0 \iff$$

$$\boxed{(\bar\ell^T H^{-1})\cdot \bar p'} = 0$$

Q.E.D.

Homographies: Basic properties



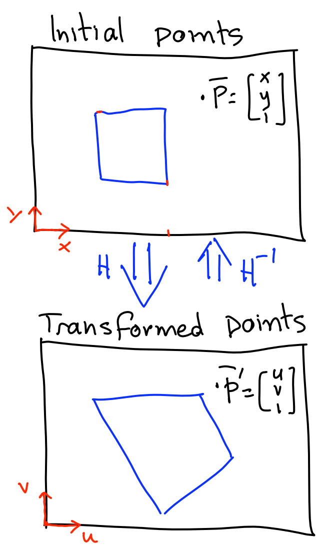
- Homographies transform lines to lines
- Scaling the homography matrix H does not affect the transformation

$$(\lambda \cdot H)\bar{p} = H \cdot (\lambda \bar{p}) \cong H\bar{p}$$

- H is easy to go back & forth between the original & transformed points

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \cong H \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cong H^{-1} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Homographies: Basic properties



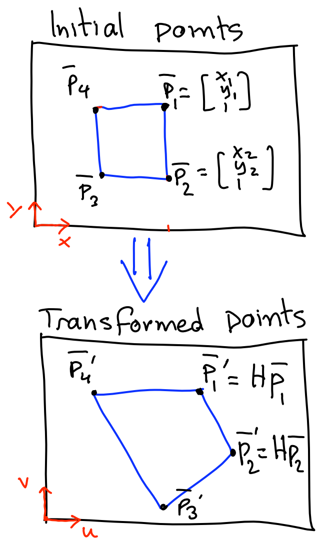
- Homographies are associative

$$H_2\left(H_1\bar{p}\right) = \left(H_2 H_1\right)\bar{p}$$

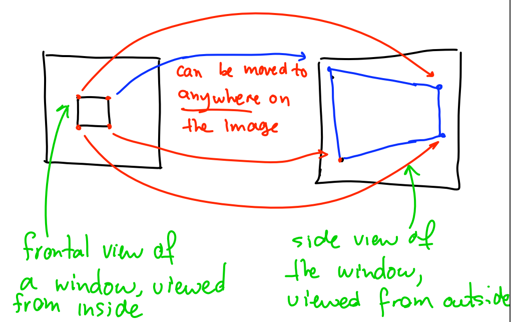
- Homographies are <u>not</u> commutative in general

$$H_2(H_1\bar{p}) \neq H_1(H_2\bar{p})$$

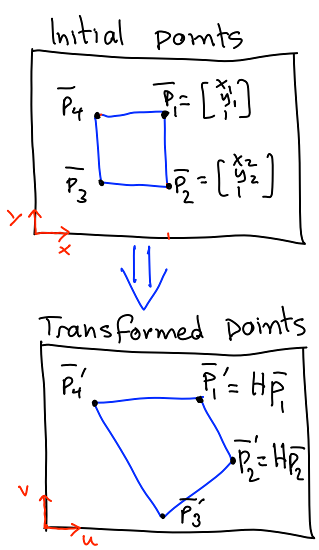
Homographies: Geometric Intuition



Linear warps correspond to every possible distortion of a square created by moving its vertices to arbitrary locations

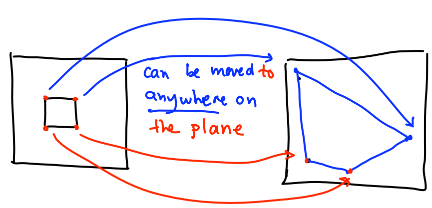


Homographies from Point Correspondences

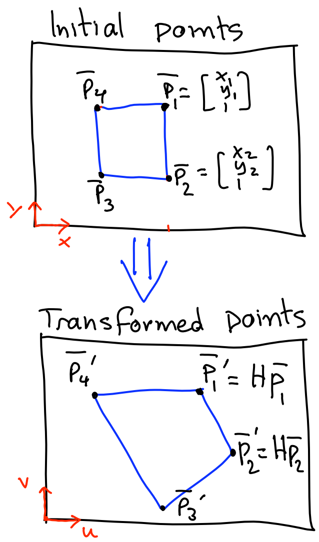


- Intuition:

  If we have a correspondence between 4 points in the two images, we can compute H



Homographies from Point Correspondences



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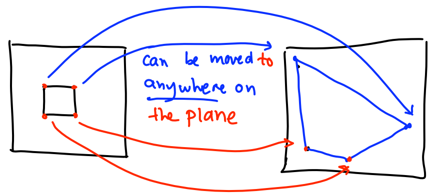
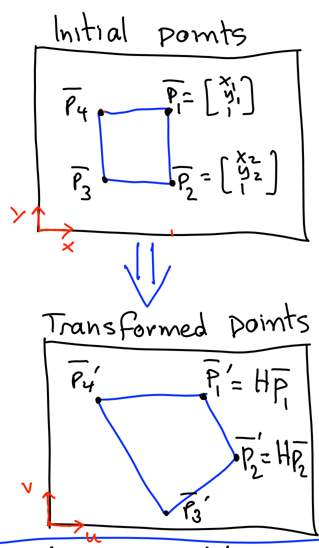
  If we have a correspondence between 4 points in the two images, we can compute H:



$$\underbrace{\begin{bmatrix}u\\v\\1\end{bmatrix}}_{\text{known}} \cong \underbrace{\overset{H}{\begin{bmatrix}a&b&c\\d&e&f\\h&i&j\end{bmatrix}}}_{\text{unknown}} \underbrace{\begin{bmatrix}x\\y\\1\end{bmatrix}}_{\text{known}}$$

Homographies from Point Correspondences

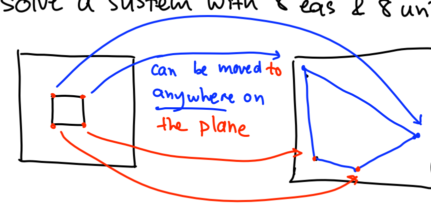


$$\boxed{u_k = (a x_k + b y_k + c)/(h x_k + k y_k + \ell) \Leftrightarrow u_k(h x_k + k y_k + \ell) - (a x_k + b y_k + c) = 0}$$

1. Each correspondence gives 2 linear equations in the 9 unknowns (so 4 correspondences ⇒ 8 eqs, 9 unknowns)

$$\boxed{\begin{aligned} a x_k + b y_k + c - u_k(h x_k + k y_k + \cancelto{1}{\ell}) &= 0 \\ d x_k + e y_k + f - v_k(h x_k + k y_k + \cancelto{1}{\ell}) &= 0 \end{aligned}}$$

2. Since any multiple of H will do, we pick one element and set it to one (e.g. $\ell = 1$) & solve a system with 8 eqs & 8 unknowns



$$\underbrace{\begin{bmatrix}u_k\\v_k\\1\end{bmatrix}}_{\text{known}} \cong \underbrace{\overset{H}{\begin{bmatrix}a&b&c\\d&e&f\\h&k&\cancelto{1}{\ell}\end{bmatrix}}}_{\text{unknown}} \underbrace{\begin{bmatrix}x_k\\y_k\\1\end{bmatrix}}_{\text{known}}$$

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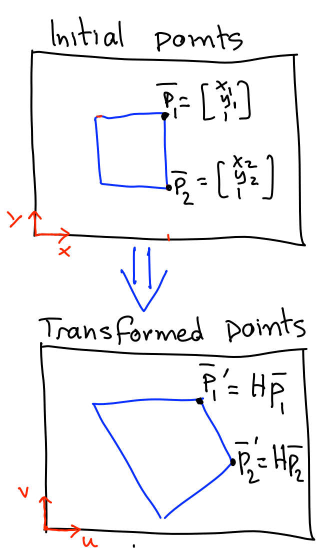
General Linear 2D Transformations



- Homographies represent a very general set of transformations

General linear (preserve lines)

Affine (preserve parallelism)

- Arbitrary shearing
- General scaling

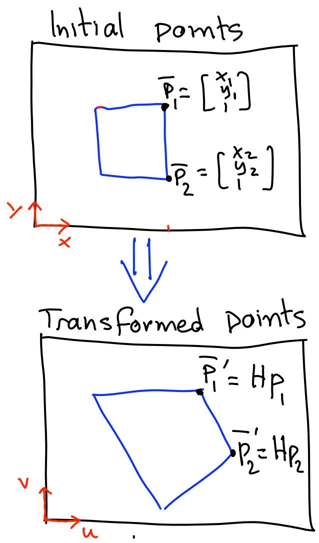
Conformal (preserve angles)

- Uniform scaling

Rigid (preserve lengths)

- Translation
- Rotation

Affine Transformations



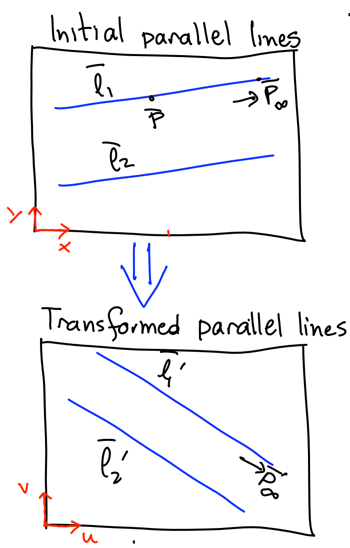
- Homographies represent a very general set of transformations

General linear (preserve lines)

Affine (preserve parallelism)

The matrix H now takes a more restricted form!

Affine Transformations: General Matrix Form



What form should H take to preserve parallel lines?

- $\bar{l}_1, \bar{l}_2$ parallel

$\Leftrightarrow$

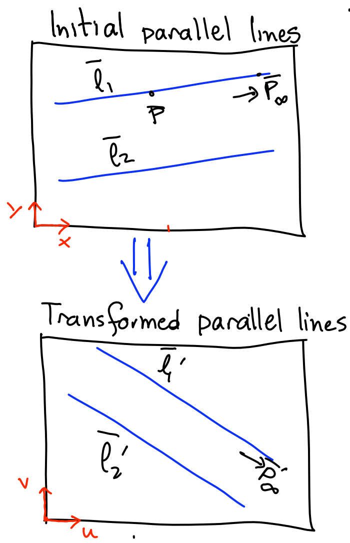
their intersection is a point $\bar{p}_\infty$ which lies "at infinity"

$\Leftrightarrow$

$\bar{p}_\infty$ must have 3rd coordinate 0

$$\bar{p}_\infty = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Affine Transformations: General Matrix Form



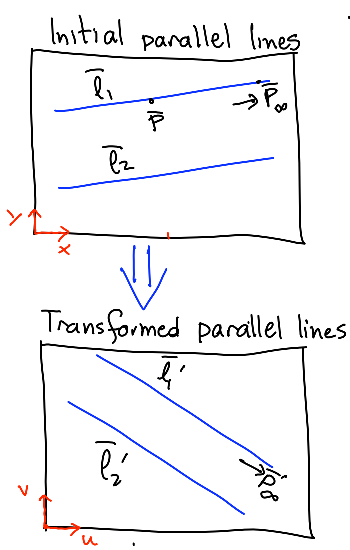
What form should H take to preserve parallel lines?

- $\bar{l}_1, \bar{l}_2$ parallel $\Leftrightarrow \bar{p}_\infty = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
- $\bar{l}_1', \bar{l}_2'$ parallel $\Leftrightarrow \bar{p}_\infty' = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$

Therefore H must satisfy

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \underbrace{\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\overset{\text{arbitrary 2x2 matrix}}{A}} & \overset{\text{2x1 vector}}{\vec{t}} \\ \hline 0 & 0 & 1 \end{array}\right]}_{\text{3x3 matrix}} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Affine Transformations: General Matrix Form



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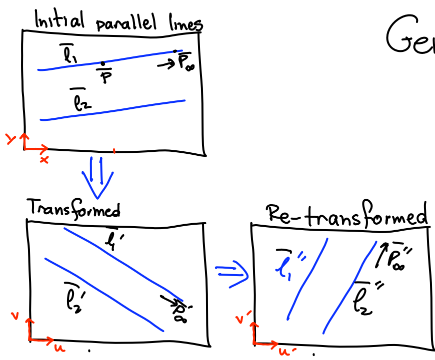
Affine Transformations: Basic Properties



General form of matrix H

arbitrary 2x2 matrix

2x1 vector

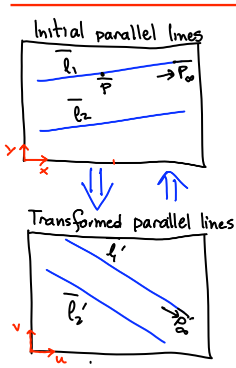
$$H = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \bar{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

1. Affine transforms are closed under composition:

this is an affine transformation!

$$\left[\begin{array}{c|c} A_2 & \bar{t}_2 \\ \hline 0\ 0 & 1 \end{array}\right]\left[\begin{array}{c|c} A_1 & \bar{t}_1 \\ \hline 0\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} A_2A_1 + \bar{t}_2[0\ 0] & A_2\cdot\bar{t}_1 + \bar{t}_2\, 1 \\ \hline [0\ 0]A_1 + 1\cdot[0\ 0] & [0\ 0]\bar{t}_1 + 1\cdot 1 \end{array}\right] = \left[\begin{array}{c|c} A_2A_1 & A_2\bar{t}_1 + \bar{t}_2 \\ \hline 0\ 0 & 1 \end{array}\right]$$

Affine Transformations: Basic Properties



General form of matrix H

arbitrary 2x2 matrix

2x1 vector

$$H = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \bar{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

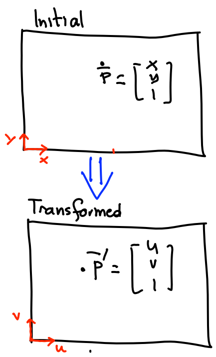
1. Affine transforms are closed under composition:

If H_1, H_2 are affine transform matrices, so is $H_1 \cdot H_2$.

2. The inverse H^{-1} of an affine transform H is affine

Proof: By definition, H^{-1} will map $\bar{P}'_\infty$ to $\bar{P}_\infty$. Since it preserves points at infinity, its matrix must have the above form. QED.

Affine Transformations: Basic Properties



General form of matrix H

arbitrary 2x2 matrix

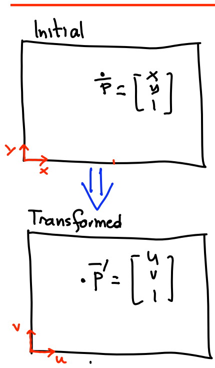
2x1 vector

$$H = \left[\begin{array}{c:c} A & \vec{t} \\ \hdashline 0\ 0 & 1 \end{array}\right]$$

3. Affine transforms preserve the value of the last homogeneous coordinate

$$\left[\begin{array}{c:c} A & \vec{t} \\ \hdashline 0\ 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ \hdashline 1 \end{array}\right] = \begin{bmatrix} A\begin{bmatrix} x \\ y \end{bmatrix} + \vec{t} \\ [0\ 0]\begin{bmatrix} x \\ y \end{bmatrix} + 1 \end{bmatrix} = \begin{bmatrix} A\begin{bmatrix} x \\ y \end{bmatrix} + \vec{t} \\ 1 \end{bmatrix}$$

Affine-Transforming 2D Points



General form of matrix H

arbitrary 2x2 matrix

2x1 vector

$$H = \left[\begin{array}{c:c} A & \vec{t} \\ \hdashline 0\ 0 & 1 \end{array}\right]$$

3. Affine transforms preserve the value of the last homogeneous coordinate

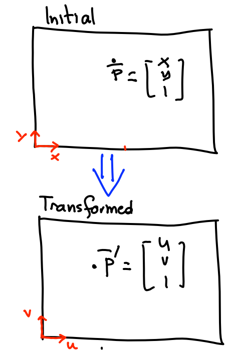
becomes equality

$$\left[\begin{array}{c:c} A & \vec{t} \\ \hdashline 0\ 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ \hdashline 1 \end{array}\right] = \begin{bmatrix} A\begin{bmatrix} x \\ y \end{bmatrix} + \vec{t} \\ 1 \end{bmatrix}$$

$\Rightarrow$ Transforming Euclidean points: (i.e. non-homogeneous)

1. append a 3^{rd} coord of 1
   $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
2. apply H:
   $\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = H \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
3. delete 3^{rd} coord of result

Affine-Transforming 2D Points



$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{H} \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \bar{t}$$

General form of matrix H

arbitrary 2x2 matrix

2x1 vector

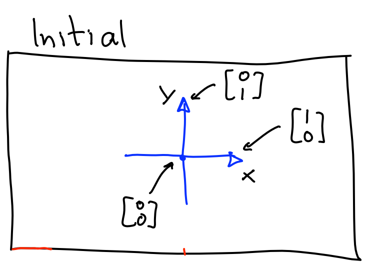
$$H = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \bar{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

Transforming Euclidean points:
(i.e. non-homogeneous)

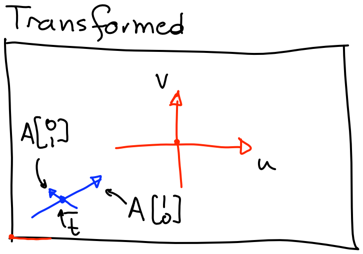
1. append a 3^{rd} coord of 1
   $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
2. apply H:
   $\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = H \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
3. delete 3^{rd} coord of result

Affine-Transforming 2D Points

Initial



Transformed



General form of matrix H

arbitrary 2x2 matrix

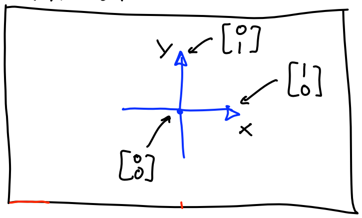
2x1 vector

$$H = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \bar{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{H} \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \bar{t}$$

Geometric Interpretation of Affine Matrix

Initial



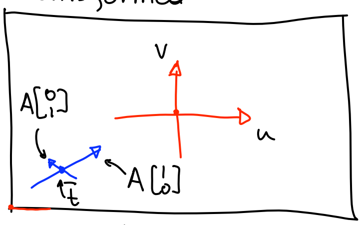
General form of matrix H

arbitrary 2x2 matrix

2x1 vector

$$H = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \vec{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

Transformed



$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\c\end{bmatrix}$$

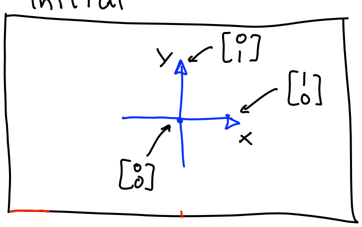
$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}b\\d\end{bmatrix}$$

1^{st} column of $A \Rightarrow$ transforms the x-axis

2^{nd} column of $A \Rightarrow$ transforms the y-axis

Geometric Interpretation of Affine Matrix

Initial



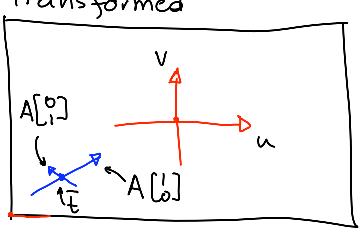
General form of matrix H

arbitrary 2x2 matrix

2x1 vector

$$H = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \vec{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

Transformed



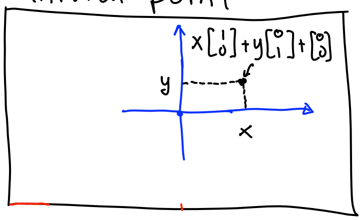
$\vec{t} \Rightarrow$ translates the origin to point $\vec{t}$

1^{st} column of $A \Rightarrow$ transforms the x-axis

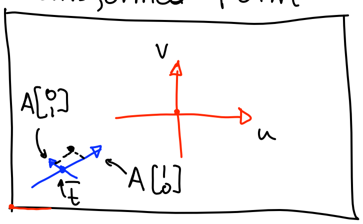
2^{nd} column of $A \Rightarrow$ transforms the y-axis

Geometric Interpretation of Affine Matrix

Initial point



Transformed Point



General form of matrix H

arbitrary 2x2 matrix

2x1 vector

$$H = \left[\begin{array}{cc|c} & A & \bar{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

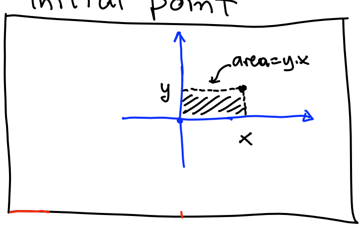
$\bar{t} \Rightarrow$ translates the origin to point $\bar{t}$

1st column of A $\Rightarrow$ transforms the x-axis

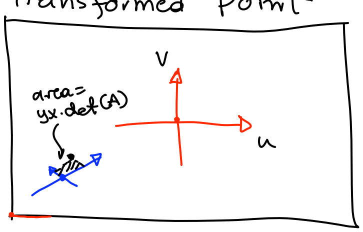
2nd column of A $\Rightarrow$ transforms the y-axis

How Affine Transformations Affect Area

Initial point



Transformed Point



General form of matrix H

arbitrary 2x2 matrix

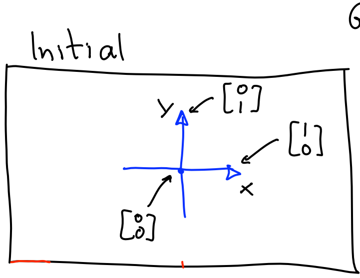
2x1 vector

$$H = \left[\begin{array}{cc|c} & A & \bar{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

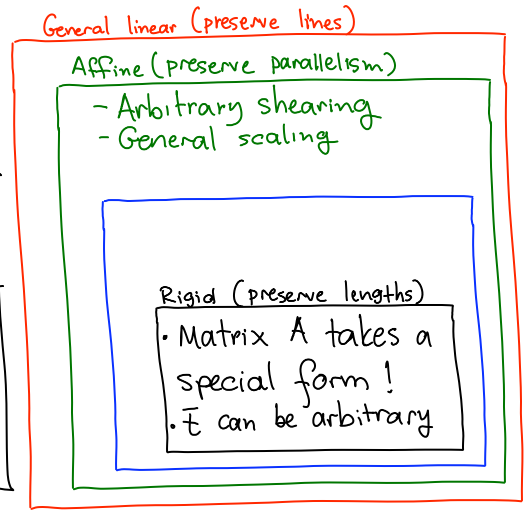
The area of any closed region will be multiplied by det(A) after the transformation $\boxed{\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc}$

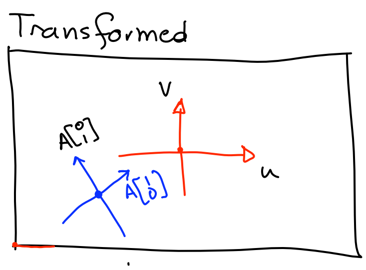
$\Rightarrow$ If A singular (non-invertible) region is 'squashed' to zero

From Affine to Rigid Transformations

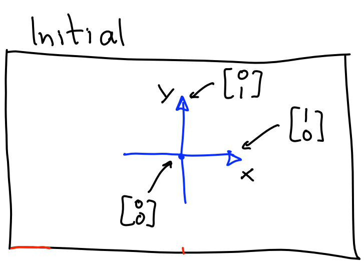


General: $\left[\begin{array}{c:c} A & \bar{t} \\ \hdashline \alpha\ \beta & 1 \end{array}\right]$ Affine: $\left[\begin{array}{c:c} A & \bar{t} \\ \hdashline 0\ 0 & 1 \end{array}\right]$





Rigid Transformations: Rotations



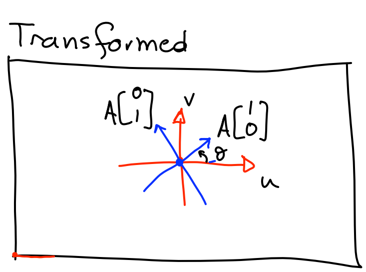
- Rigid (a.k.a "Euclidean") transforms are composed of 2D rotations and/or translations $\left[\begin{array}{c:c} A & \bar{t} \\ \hdashline 0\ 0 & 1 \end{array}\right]$
- Rotation by a counter-clockwise angle θ: about origin

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

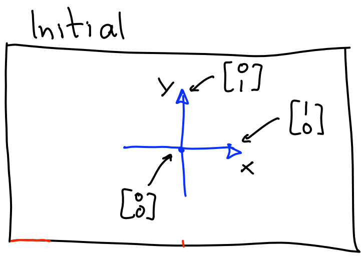
- "Pure" rotation transformation

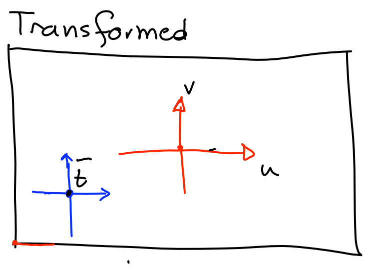
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

no translation of origin



Rigid Transformations: Translations





- Rigid (a.k.a "Euclidean") transforms are composed of 2D rotations and/or translations $\begin{bmatrix} A & \bar{t} \\ 0\;0 & 1 \end{bmatrix}$

- Translation by vector $\bar{t}$

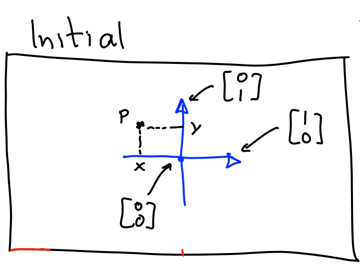
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

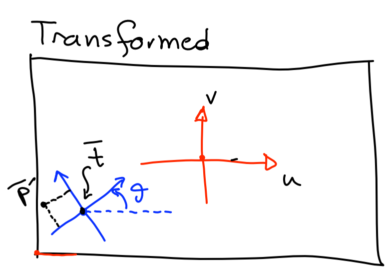
- "Pure" translation transformation

no rotation

$$\begin{bmatrix} 1 & 0 & \bar{t} \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

Composition of 2D Rotations & Translations





Example 1: Rotation followed by translation

- First rotate about θ,

$$\bar{p} \rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{p}$$

- Then translate the result by vector $\bar{t}$:

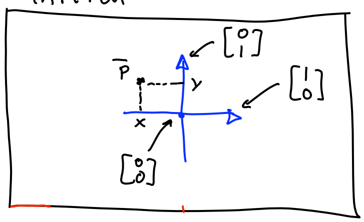
$$\bar{p}' = \begin{bmatrix} 1 & 0 & \bar{t} \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{p}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & \bar{t} \\ \sin\theta & \cos\theta & \\ 0 & 0 & 1 \end{bmatrix} \bar{p}$$

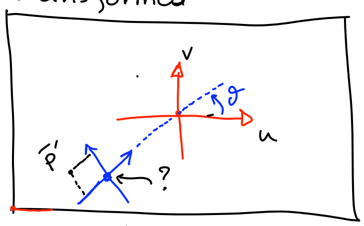
Composition of 2D Translations & Rotations

Example 2: Translation followed by rotation

Initial



Transformed



- First translate by $\bar{t}$

$$\bar{p} \rightarrow \begin{bmatrix}1 & 0 & \bar{t} \\ 0 & 1 & \\ 0 & 0 & 1\end{bmatrix} \bar{p}$$

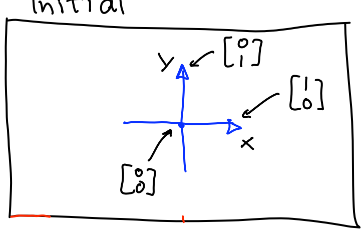
- Then rotate the result by θ

$$\bar{p}' = \begin{bmatrix}\cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & \bar{t} \\ 0 & 1 & \\ 0 & 0 & 1\end{bmatrix} \bar{p}$$

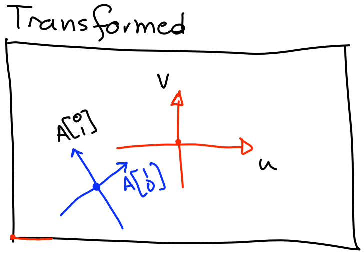
$$= ?$$

From Affine to Conformal Transformations

Initial



Transformed



Affine: $\left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \bar{t} \\ \hline 0 & 0 & 1 \end{array}\right]$

General linear (preserve lines)

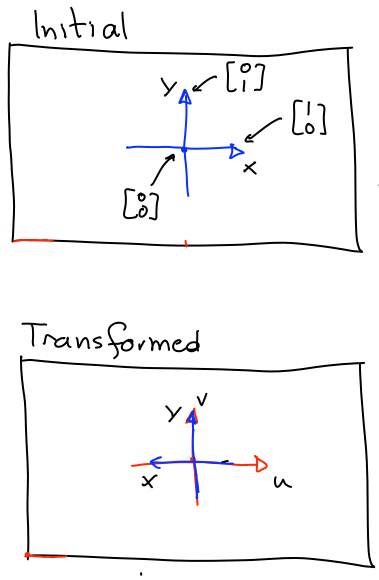
Affine (preserve parallelism)

- Arbitrary shearing
- General scaling

Conformal (preserve angles)

- Uniform scaling
- Reflection
- Translation
- Rotation

Conformal Transformations: Reflection

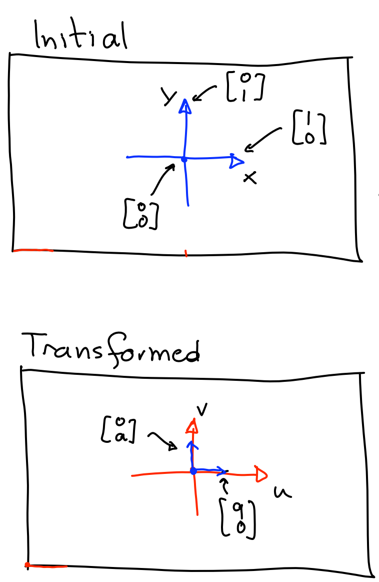


- Conformal transforms include reflections and uniform scalings $\left[\begin{array}{c|c} A & t \\ \hline 0\ 0 & 1 \end{array}\right]$

- Pure reflection (about y)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Conformal Transformations: Uniform Scaling



- Conformal transforms include reflections and uniform scalings $\left[\begin{array}{c|c} A & t \\ \hline 0\ 0 & 1 \end{array}\right]$

- Pure reflection (about y)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- "Pure" uniform scaling

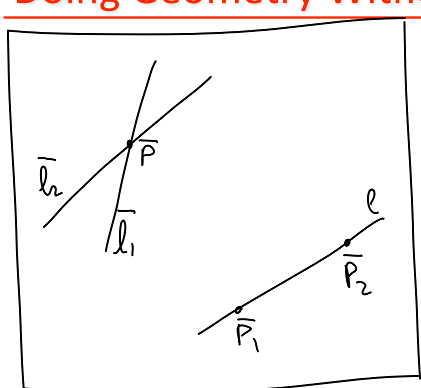
$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topic 4:

Coordinate-Free Geometry (CFG)

- A brief introduction & basic ideas

Doing Geometry Without Coordinates



- Style of expressing geometric objects & relations that avoids reliance on a coordinate system
- Useful in CG where we often deal with many coord systems

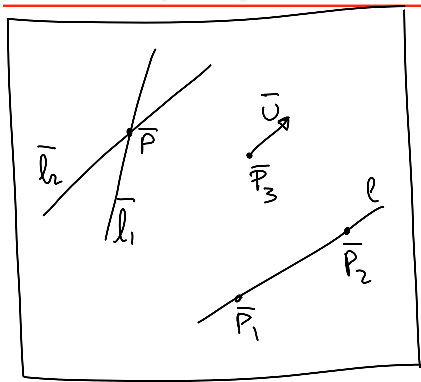
(#1) Intersection of 2 lines

$$\bar{p} = \bar{l}_1 \times \bar{l}_2$$

(#2) Line through 2 points

$$\bar{l} = \bar{p}_1 \times \bar{p}_2$$

CFG: Key Objects & their Homogeneous Repr.



Key objects:

- Points $\bar{p}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}, w \neq 0$
- Lines $\bar{\ell}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{v}$ $\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

all represented as homogeneous 3-vectors

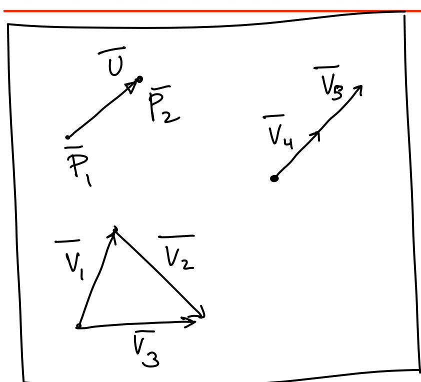
(#1) Intersection of 2 lines

$$\bar{p} = \bar{\ell}_1 \times \bar{\ell}_2$$

(#2) Line through 2 points

$$\bar{\ell} = \bar{p}_1 \times \bar{p}_2$$

CFG: Basic Geometric Operations



Key objects:

- Points $\bar{p}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}, w \neq 0$
- Lines $\bar{\ell}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{v}$ $\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

(#1) Intersection of 2 lines

$$\bar{p} = \bar{\ell}_1 \times \bar{\ell}_2$$

(#2) Line through 2 points

$$\bar{\ell} = \bar{p}_1 \times \bar{p}_2$$

(#3) Point-vector addition

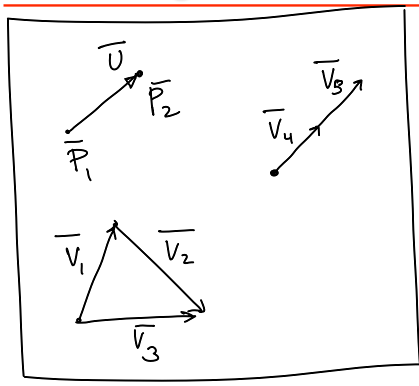
$$\bar{p}_2 = \bar{p}_1 + \bar{v}$$

(#4) Vector-vector addition

$$\bar{v}_3 = \bar{v}_1 + \bar{v}_2$$

(#5) Vector scaling: $\bar{v}_5 = \lambda \bar{v}_4$

CFG: "Legal" vs. "Undefined" Geometric Ops



Key objects:

- Points $\bar{p}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}, w \neq 0$
- Lines $\bar{\ell}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{U}$ $\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

(#3) Point-vector addition

$$\bar{P}_2 = \bar{P}_1 + \bar{V}$$

(#4) Vector-vector addition

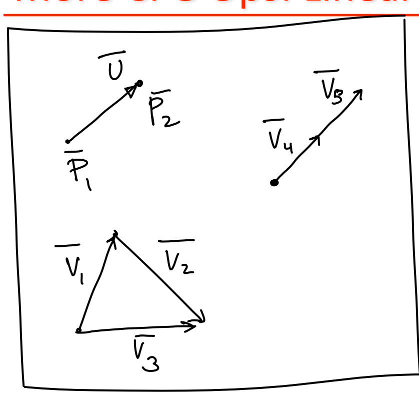
$$\bar{V}_3 = \bar{V}_1 + \bar{V}_2$$

(#5) Vector scaling: $\bar{V}_5 = \lambda \bar{V}_4$

CAUTION: Addition possible only when 3rd homogeneous coordinate not affected

e.g. $\bar{P}_1 + \bar{P}_2$ = UNDEFINED!

More CFG Ops: Linear Vector Combination



Key objects:

- Points $\bar{p}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}, w \neq 0$
- Lines $\bar{\ell}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors $\bar{U}$ $\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$

(#3) Point-vector addition

$$\bar{P}_2 = \bar{P}_1 + \bar{V}$$

(#4) Vector-vector addition

$$\bar{V}_3 = \bar{V}_1 + \bar{V}_2$$

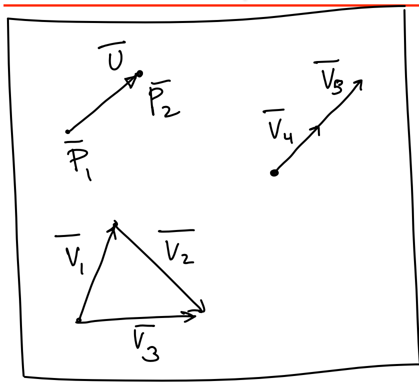
(#5) Vector scaling: $\bar{V}_5 = \lambda \bar{V}_4$

CAUTION: Addition possible only when 3rd homogeneous coordinate not affected

(#6) Linear vector combination

$$\bar{V} = \sum_{i=1}^{k} \lambda_i \bar{V}_i$$

More CFG Ops: Affine Point Combination



Key objects:

- Points $\bar p$ $\begin{bmatrix}x\\y\\w\end{bmatrix}, w\neq 0$
- Lines $\bar \ell$ $\begin{bmatrix}a\\b\\c\end{bmatrix}$
- Vectors $\bar v$ $\begin{bmatrix}\alpha\\\beta\\0\end{bmatrix}$

(#3) Point-vector addition

$$\bar P_2 = \bar P_1 + \bar v$$

(#7) Point subtraction

$$\bar v = \bar P_2 - \bar P_1$$

only when $\bar P_1, \bar P_2$ have same 3rd coord!

(#8) Affine point combination

$$\bar P = \sum_{i=1}^{K} \lambda_i \bar P_i$$

only when all $\bar P_i$ have same 3rd coord AND

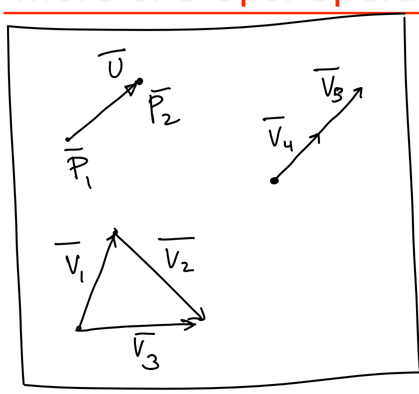
- $\sum_{i=1}^{K} \lambda_i = 1$. i.e. circled expression is a vector ⇒ reduces to (#3)

OR

- $\sum_{i=1}^{K} \lambda_i = 0$ i.e. circled expression is a point $\bar q$ with same 3rd coord ⇒ reduces to (#7)

$$= \bar P_1 + (\lambda_1 - 1)\bar P_1 + \sum_{i=2}^{K} \lambda_i \bar P_i$$

More CFG Ops: Operations w/ Scalar Result



Key objects:

- Points $\bar p$ $\begin{bmatrix}x\\y\\w\end{bmatrix}, w\neq 0$
- Lines $\bar \ell$ $\begin{bmatrix}a\\b\\c\end{bmatrix}$
- Vectors $\bar v$ $\begin{bmatrix}\alpha\\\beta\\0\end{bmatrix}$

(#9) $\|\bar v\|$ magnitude of vector $\bar v$

(#10) $\bar v_1 \cdot \bar v_2$ dot product of two vectors (also written as $(\bar v_1)^T(v_2)$ in matrix notation)

(#11) $\bar \ell \cdot \bar p$ dot product of a line and a point