Today’s Topics

9. The Discrete Wavelet Transform
Topic 7:

Discrete Wavelet Transform

- Wavelet transform vs. Laplacian pyramid
- Basic intuition: a simple wavelet-like 2D transform
- The 1D Haar wavelet transform
- 1D Haar wavelet transform as a matrix product
- Reconstructing a 1D image from its wavelet coefs
- Wavelet-based image compression
- The 2D Haar wavelet transform
Wavelet-Based Image Representations

The Wavelet-based representation of images touches upon several concepts covered so far:

- Image
- Laplacian Pyramid
  - defined by a single filter
  - image represented as a pyramid of detail images
The Laplacian Pyramid Representation

• How many pixels does the Laplacian pyramid have?

\[ (2^n + 1) + (2^{n-1} + 1) + \ldots + (2^0 + 1) \]

\[ L_0 \uparrow \quad L_1 \quad \ldots \quad L_n \]

\[ 1 + \frac{1}{4} + \frac{1}{16} + \ldots \approx 4/3 \]

\[ \Rightarrow \text{Representation is overcomplete} \quad (\text{more pixels in rep than pixels in image}) \]
Wavelet-Based Image Representations

The Wavelet-based representation of images touches upon several concepts covered so far:

- Image $\Leftrightarrow$ vector in a high dimensional space

Laplacian Pyramid

defined by a single filter

image represented as a pyramid of detail images
Wavelet-Based Image Representations

The Wavelet-based representation of images touches upon several concepts covered so far:

1. Image $\Leftrightarrow$ vector in a high dimensional space

   - Laplacian Pyramid
     - defined by a single filter
     - image represented as a pyramid of detail images

   - PCA basis (eigenfaces) are an efficiently-computable & compact representation of images (from a known image class)

   - Image "coordinate" in this basis computed by a dot product
Reminder: The Eigenface/PCA Image Basis

\[ X_1 (M \text{ dimensions}) \]

\[ X_1 (d\text{-dimensional approx } d=3) \]

\[ \overline{X} \]

\[ y_1^1 + y_1^2 + y_1^3 \]

\[ B_1 \]

\[ B_2 \]

\[ B_3 \]
Representing Images by their PCA Basis

\[
\begin{bmatrix}
Z_1^1 & Z_2^1 & \cdots & Z_M^1 \\
Z_1^2 & Z_2^2 & \cdots & Z_M^2 \\
\vdots & \vdots & \ddots & \vdots \\
Z_1^N & Z_2^N & \cdots & Z_M^N
\end{bmatrix} =
\begin{bmatrix}
B_1 & B_2 & \cdots & B_M
\end{bmatrix} \cdot
\begin{bmatrix}
y_1^1 & y_2^1 & \cdots & y_N^1 \\
y_1^2 & y_2^2 & \cdots & y_N^2 \\
\vdots & \vdots & \ddots & \vdots \\
y_1^M & y_2^M & \cdots & y_N^M
\end{bmatrix}
\]

- Image reconstruction:
  \[
  [x_i^1] = B \cdot [y_i^1] + \bar{X}
  \]
- Image transform:
  \[
  [y_i] = B^T \cdot [x_i - \bar{X}]
  \]

\[G_{12} = 0 \text{ in this basis}\]
\[G_{12} \gg 0 \text{ in this basis}\]
The Discrete Wavelet Transform

Image reconstruction:

1. \[ \begin{bmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_M^i \end{bmatrix} = B \cdot \begin{bmatrix} y_1^i \\ y_2^i \\ \vdots \\ y_M^i \end{bmatrix} + \overline{X} \]

Image transform

2. \[ \begin{bmatrix} y_1^i \\ y_2^i \\ \vdots \\ y_M^i \end{bmatrix} = B^T \left[ x_i - \overline{X} \right] \]

The (discrete) wavelet transform maps an image into yet another basis defined by a "special" matrix \( B \):

- captures scale
- invertible, orthogonal, square
- image independent
The Discrete Wavelet Transform

Image reconstruction:
1. \[
\begin{bmatrix}
\bar{x}_1' \\
\bar{x}_2' \\
\vdots \\
\bar{x}_m'
\end{bmatrix} = B \cdot \begin{bmatrix}
y_1' \\
y_2' \\
\vdots \\
y_m'
\end{bmatrix} + \bar{x}
\]
Image transform
2. \[
\begin{bmatrix}
y_1' \\
y_2' \\
\vdots \\
y_m'
\end{bmatrix} = B^T \begin{bmatrix}
x_1 - \bar{x} \\
x_2 - \bar{x} \\
\vdots \\
x_m - \bar{x}
\end{bmatrix}
\]

The (discrete) wavelet transform maps an image into yet another basis defined by a "special" matrix \( B \)
- captures scale
- invertible, orthogonal, square
- image independent
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A Simple, Minimal 2-D Image Transform

Properties of transformation
- Minimal (no "wasted" pixels)
- Multiple scales represented simultaneously
- Invertible, linear

Input image \( (2^N \times 2^N) \)

Transformed image \( (2^N \times 2^N) \)

Wavelet transform
A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1} \times 2^{N-1}$ as shown in figure.

Input image ($2^N \times 2^N$)

Transformed image ($2^N \times 2^N$)
A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1} \times 2^{N-1}$ as shown in figure.
A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1} \times 2^{N-1}$ as shown in figure.

Input image ($2^N \times 2^N$)

Transformed image ($2^N \times 2^N$)

$W_0$: 

$A = \frac{1}{4} (p_1 + p_2 + p_3 + p_4)$
A Simple, Minimal 2-D Image Transform

Step 2: Recursively perform Step 1 for top-left quadrant of result

Result of Step 1 ($2^{N-1} \times 2^{N-1}$)

Transformed image ($2^N \times 2^N$)
A Simple, Minimal 2-D Image Transform

Step 3: Recursion stops when average image is 1 pixel

Transformed image \((2^N \times 2^N)\)
Invertibility of the Transformation

Property: $W_k$ can be reconstructed from $W_{k+1}$

$p_1 = 4p_2 - p_3 - p_4$

$W_k \iff W_0$ reconstructible from $W_n$
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1D Haar Wavelet Transform: Recursive Definition

Input image $I^3$

<table>
<thead>
<tr>
<th>Pixel #</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$2^3-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

Average $I^2$

<table>
<thead>
<tr>
<th>Average $I^2$</th>
<th>8</th>
<th>4</th>
<th>8</th>
<th>4</th>
</tr>
</thead>
</table>

take average of first two pixels

difference of first pixel from the average

Detail coefs $D^2$

<table>
<thead>
<tr>
<th>Detail Coefs $D^2$</th>
<th>1</th>
<th>-1</th>
<th>-2</th>
<th>-2</th>
</tr>
</thead>
</table>

(analogous to level $L_0$ of Gaussian pyramid)

(note: we don’t need to store difference of 2nd pixel from average $\Rightarrow$

$D^0$ has $\frac{1}{2}$ the size of the corresponding Laplacian level $L_0$ !)
1D Haar Wavelet Transform: Recursive Definition

Input image \( I^3 \):

\[
\begin{array}{ccccccccc}
0 & 1 & 2 & \ldots & 2^{3-1} \\
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6
\end{array}
\]

Average \( I^2 \):

\[
\begin{array}{cccc}
8 & 4 & 8 & 4 \\
1 & -1 & -2 & -2
\end{array}
\]

Detail coefs \( D^2 \)

Average \( I^1 \):

\[
\begin{array}{cccc}
6 & 6 \\
2 & 2
\end{array}
\]

Detail coefs \( D^1 \)

\[
I_i^j = \frac{1}{2} \left( I_{2i}^{j-1} + I_{2i+1}^{j-1} \right)
\]

\( j \)-th level of "pyramid" contains \( 2^j \) pixels

\[
D_i^j = I_{2i}^{j-1} - \frac{1}{2} \left( I_{2i}^{j-1} + I_{2i+1}^{j-1} \right)
\]

\[
= \frac{1}{2} \left( I_{2i}^{j-1} - I_{2i+1}^{j-1} \right)
\]
1D Haar Wavelet Transform: Recursive Definition

Input image $I^3$:

<table>
<thead>
<tr>
<th>Pixel #</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$2^3-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

Average $I^2$:

$8 \ 4 \ 4 \ 4$ | $1 \ -1 \ -2 \ -2$

Average $I^1$:

$6 \ 6$ | $2 \ 2$

equivalent Convolution mask $\phi$:

$\frac{1}{2} \ \frac{1}{2}$

$I^{j}_i = \frac{1}{2} \left( I^{j+1}_{2i} + I^{j+1}_{2i+1} \right)$

$j$-th level of “pyramid” contains $2^j$ pixels

equivalent Convolution mask $\Phi$:

$-\frac{1}{2} \ \frac{1}{2}$

$D^j = \frac{1}{2} \left( I^{j+1}_{2i} - I^{j+1}_{2i+1} \right)$

Detail coefs $D^2$

Detail coefs $D^1$
1D Haar Wavelet Transform: Recursive Definition

Input image $I^3$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$2^3-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Average $I^2$

$| 8 | 4 | 8 | 4 |

Detail coefs $D^2$

Average $I^1$

$| 6 | 6 |

Detail coefs $D^1$

Average $I^0$

$| 6 |

Detail coefs $D^0$

$$I_i^j = \frac{1}{2} \left( I_{2i}^{j+1} + I_{2i+1}^{j+1} \right)$$

$j$-th level of "pyramid" contains $2^j$ pixels

$$D_i^j = \frac{1}{2} \left( I_{2i}^{j+1} - I_{2i+1}^{j+1} \right)$$
1D Haar Wavelet Transform: Recursive Definition

Input image

<table>
<thead>
<tr>
<th>pixel #</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>2 - 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>2</td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Wavelet transform contains these pixels

Wavelet transform coefs

Detail coefs $D^2$
Detail coefs $D'$
Detail coefs $D$

Wavelet-transformed image

$I^0$, $D^0$, $D^1$, $D^2$

<table>
<thead>
<tr>
<th>scale 0</th>
<th>scale 1</th>
<th>scale 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>-2</td>
</tr>
</tbody>
</table>
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1D Haar Wavelet Transform as a Matrix Product

Input image:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>2^n-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

Wavelet transformed image:

\[ I^0 = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}, \quad D^0 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}, \quad D^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[
\frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
1D Haar Wavelet Transform as a Matrix Product

Input image:

\[
\begin{bmatrix}
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6
\end{bmatrix}
\]

Wavelet transformed image:

\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Original image:

\[
\begin{bmatrix}
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6
\end{bmatrix}
\]
1D Haar Wavelet Transform as a Matrix Product

Input image:

\[
\begin{bmatrix}
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6 \\
\end{bmatrix}
\]

Matrix product:

\[
\begin{bmatrix}
6 & 0 \\
2 & 2 \\
2 & 1 \\
1 & -1 \\
-2 & -2 \\
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
8 \\
4 \\
8 \\
4 \\
-1 \\
-2 \\
-2 \\
\end{bmatrix}
\]
1D Haar Wavelet Transform as a Matrix Product

Input image:

\[
\begin{bmatrix}
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6
\end{bmatrix}
\]

Matrix product:

\[
\begin{pmatrix}
2 & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
I_0 \\
D_0 \\
D_1 \\
D_2
\end{pmatrix}
= \\
\begin{pmatrix}
6 & 0 \\
2 & 2 \\
1 & -1 \\
-2 & -2
\end{pmatrix}
= \\
\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\cdot \\
\frac{1}{2}
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
9 \\
7 \\
3 \\
5 \\
6 \\
10 \\
2 \\
6
\end{pmatrix}
\]
1D Haar Wavelet Transform as a Matrix Product

3rd & 4th rows of product

\[
\begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}
\end{bmatrix}
\]
1D Haar Wavelet Transform as a Matrix Product

Input image:

\[
\begin{array}{cccccccc}
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6 \\
\end{array}
\]

Wavelet transformed image:

\[
\begin{bmatrix}
1 & 2 & 0 & 2 \\
2 & 2 & 1 & -1 \\
-2 & -2 & 1 & 1 \\
-2 & -2 & 1 & 1 \\
\end{bmatrix}
\]

Original image:

\[
\begin{bmatrix}
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6 \\
\end{bmatrix}
\]
1D Haar Wavelet Transform as a Matrix Product

Input image

\[
\begin{array}{ccccccccc}
0 & 1 & 2 & \ldots & 2^n-1 \\
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6
\end{array}
\]

Wavelet transformed image

\[
\begin{bmatrix}
I^0 \\
D^0 \\
D^1 \\
D^2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
9 & 7 & 3 & 5 & 6 & 10 & 2 & 6
\end{bmatrix}
\]

Original image
The 1D Haar Wavelet Transform Matrix $W$

- Matrix contains $N-1$ scales.
- Scale $j$ represented by $2^j$ rows: $\psi_0^j, \cdots, \psi_{2^j-1}^j$.

Row $\psi_i^j$ has $\frac{2^n}{2^j} = 2^{n-j}$ non-zero pixels.
- They are pixels $x = i 2^{n-j}, \cdots, (i+1)2^{n-j} - 1$ with $|\psi_i^j(x)| = \frac{1}{2^{n-j}}$.

Wavelet transformed image $D'$:

\[
\begin{bmatrix}
6 \\
0 \\
2 \\
2 \\
1 \\
-1 \\
-2 \\
-2 \\
\end{bmatrix}
\]

Original image:

\[
\begin{bmatrix}
9 \\
7 \\
3 \\
5 \\
6 \\
10 \\
2 \\
6 \\
\end{bmatrix}
\]
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Reconstructing an Image from its Wavelet Coefs

Question: What is the dot product of two distinct rows of W?

\[ W = \begin{bmatrix} \psi_0 \\ \vdots \\ \psi_{j \epsilon I} \end{bmatrix} \]

\[
\begin{bmatrix}
I^0 \\
D^0 \\
D^1 \\
D^2 \\
\end{bmatrix} \begin{bmatrix}
6 \\
0 \\
2 \\
2 \\
\end{bmatrix} = \begin{bmatrix}
9 \\
7 \\
3 \\
5 \\
\end{bmatrix} \]

Original image
Reconstructing an Image from its Wavelet Coefs

Answer:

\[ \psi_i^j \cdot \psi_i^{j'} = 0 \]

for two distinct rows of \( W \iff W W^T = \text{diagonal} \)

Wavelet transformed image:

\[ \begin{bmatrix}
  I_0 & 6 \\
  D_0 & 0 \\
  D_1 & 2 \\
  D_2 & -1 \\
  D_2 & -2 \\
\end{bmatrix} = \begin{bmatrix}
  9 \\
  7 \\
  3 \\
  5 \\
  6 \\
  10 \\
  2 \\
  6 \\
\end{bmatrix} \]

Original image
Reconstructing an Image from its Wavelet Coefs

\[
WW^T = \text{diagonal} \quad \frac{1}{2^{n-j}}
\]

\[
(\psi_i^j)(\psi_i^j)^T = \begin{bmatrix}
\end{bmatrix}
\]

\[
W = \begin{bmatrix}
\frac{1}{2^n}
\psi_0
\vdots
\psi_{2^{n-1}}
\end{bmatrix}
\]

Original image

\[
\begin{bmatrix}
9
7
3
5
6
10
2
6
\end{bmatrix}
\]

Wavelet transformed image

\[
\begin{bmatrix}
I_0
D_0
D_1
D_2
\end{bmatrix}
\begin{bmatrix}
6
0
2
1
-1
-2
-2
\end{bmatrix}
\]
Reconstructing an Image from its Wavelet Coefs

Define \( \Lambda = W W^T \)

with

\[
\Lambda = \begin{bmatrix}
\otimes & 0 \\
0 & \mathbb{I}
\end{bmatrix}
\]

of the form \( \frac{1}{2^{n-j}} \)

Multiply \( W^T \) on both sides:

\[
W^T \cdot \begin{bmatrix}
6 \\
0 \\
2 \\
-2 \\
-1 \\
-1 \\
-2 \\
-2
\end{bmatrix} = W^T \cdot \begin{bmatrix}
9 \\
7 \\
3 \\
5 \\
6 \\
10 \\
2 \\
6
\end{bmatrix}
\]

Original image
Reconstructing an Image from its Wavelet Coefs

Define $\Lambda = WW^T$

with $\Lambda = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 2^{-n} \end{bmatrix}$

of the form $\frac{1}{2^{n-j}}$

$W = \begin{bmatrix} 2^{j-n} & -2^{j-n} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$

Multiply $\Lambda^{-1} W^T$ on both sides:

$\Lambda^{-1} W^T \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ -1 \\ -1 \\ 0 \\ 2 \\ -2 \\ -2 \end{bmatrix} = \Lambda^{-1} W^T \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$

Original Image
Reconstructing an Image from its Wavelet Coefs

Observation:
\[ \Lambda^{-1/2} W \text{ is orthogonal because} \]
\[ (\Lambda^{-1/2} W)(\Lambda^{1/2} W)^T = \]
\[ \Lambda^{-1/2} W W^T \Lambda^{-1/2} = \]
\[ \Lambda^{-1/2} \Lambda \Lambda^{-1/2} = I \]

Therefore
\[ \Lambda^{-1} W^T \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \]

\[ W = \begin{bmatrix} 2^{1-n} - 2^{1-n} \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \]

Original Image
\[ \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix} \]
Reconstructing an Image from its Wavelet Coefs

\[ \Lambda^{-1} = \begin{bmatrix} a_0^{-1} & 0 \\ 0 & 2^{-N} \end{bmatrix} \]

So we have

\[ \Lambda^{-1} \]

\[ W = \begin{bmatrix} 2^{1-N} & -2^{1-N} \\ \text{zero} & \frac{2^N}{2} \end{bmatrix} \]

\[ \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix} \]

Original image
Interpreting the Wavelet Coefficients

\[ I = \frac{a_0^{-1}}{a_i^{-1}} \]

\[ W = \begin{bmatrix} 2^{j-1} & -2^{j-2} \\ \text{zero} & 2^{j-1} \end{bmatrix} \]

\[ \begin{bmatrix} a_0^{1} \\ \vdots \\ a_0^{n-1} \\ 0 \\ 0 \\ \vdots \\ a_{2^n-1}^{1} \\ a_{2^n-1}^{2} \\ \vdots \end{bmatrix} \]

\[ \begin{bmatrix} 6 & 0 & 2 & 2 \\ -1 & -1 & 1 & 0 \\ -2 & -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 1 \text{0} \\ 2 \\ 6 \end{bmatrix} \]

Original image
Interpreting the Wavelet Coefficients

By multiplying $I$ with $W$ we obtain a decomposition of the image into a sequence of basis images $\psi^0, \psi^1, \ldots \psi^j, \ldots$ that form an orthogonal basis of $\mathbb{R}^2$.
The wavelet coefficients are the coordinates of the image, considered as a vector in \( \mathbb{R}^{2^n} \), in the basis defined by images \( \phi_0, \psi_0, \psi', \ldots \).
The Normalized Haar Wavelet Matrix

We can normalize the wavelet transform matrix by multiplying

\[ \tilde{W} = \left[ \sqrt{a_1}, \ldots, \sqrt{a_{2^n-1}} \right] \cdot W \]

\[ \tilde{W} = \begin{bmatrix} 9 \end{bmatrix} \]

Original image

\[ \begin{bmatrix} 7 & 3 & 5 & 6 & 1 & 0 & 2 & 6 \end{bmatrix} \]
The Normalized Haar Wavelet Coefficients

\[ I = \begin{array}{c}
C_0^0 \times \\
d_0^0 \times \\
da_0^1 \times \\
da_1^1 \times \\
da_2^1 \times \\
da_2^2 \times \\
da_2^2 \times \\
da_2^3 \times \\
\end{array} \quad \begin{array}{c}
\psi_0^0 \vdash \\
\psi_0^1 \\
\psi_1^1 \\
\psi_1^2 \\
\psi_1^3 \\
\psi_1^4 \\
\psi_1^5 \\
\end{array} \]

\[ \tilde{W} = \]

By multiplying \( I \) with \( \tilde{W} \) we obtain a set of wavelet coefficients \( c_0, d_0, ... \) that express \( I \) as a linear combination of the basis images \( \phi_0, \psi_0, \psi_1, \psi_2, ... \)
Topic 7:

Discrete Wavelet Transform

- Wavelet transform vs. Laplacian pyramid
- Basic intuition: a simple wavelet-like 2D transform
- The 1D Haar wavelet transform
- 1D Haar wavelet transform as a matrix product
- Reconstructing a 1D image from its wavelet coefs
- Wavelet-based image compression
- The 2D Haar wavelet transform
Wavelet Compression Algorithm #1

Input: 1D image $I$, compression $k$
Output: $k \cdot 2^n$ coefficients

1. Compute $\tilde{W} I$
2. Sort the coefficients $c_0, d_0, d_1, \ldots$ in order of decreasing absolute value
3. Keep the top $k \cdot 2^n$ coeffs

$\tilde{W} = \begin{bmatrix} c_0 \quad d_0 \quad d_1 \quad \ldots \end{bmatrix}$

*Readings show that the algorithm gives the best least-squares approx of the image for the given compression level

Original image $I$

$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\
2 \\ 6 \end{bmatrix}$
Wavelet Compression Algorithm #2

Input: 1D image \( I \), max error \( \varepsilon \)
Output: \( k \cdot 2^n \) coefficients

1. Compute \( \tilde{W} = W \overline{I} \)
2. Sort the coefficients \( c_0, d_0, d_1, \ldots \) in order of decreasing absolute value
   keep the top \( k \cdot 2^n \) coefficients with \( k \) such that
   \( |\tilde{I} - I| < \varepsilon \), where \( \tilde{I} \) is the image reconstructed from the top \( k \cdot 2^n \) coefficients

\[
\begin{bmatrix}
\tilde{c}_0 \\
\tilde{d}_0 \\
\tilde{d}_1 \\
\end{bmatrix} = \tilde{W}
\]

Original image \( I \)
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The 2D Haar Wavelet Transform

\[ I = \begin{bmatrix} I_0 & I_1 & I_2 & \ldots \end{bmatrix} \]

\[ \tilde{W} = \]

- To compute the wavelet transform of a 2D image:
  1. Compute the 1D transform for each column and place the vectors \( \tilde{W}I_i \) in a new image \( I' \)
  2. Compute the 1D transform of each row of \( I' \)
The 2D Haar Wavelet Transform

Exercise: Show that every 2D wavelet coefficient can be expressed as the result of a dot product of the image I and an image defined by \((\psi_i)^T(\psi_i')\) where \(\psi_i\) are 1D Haar basis images.

\[ \tilde{W} = \]

- To compute the wavelet transform of a 2D image:
  1. Compute the 1D transform for each column and place the vectors \(\tilde{W}_i\) in a new image \(I'\).
  2. Compute the 1D transform of each row of \(I'\).
The 2-D Haar Wavelet Basis

Definition of the first few (coarsest scale) wavelet coefficients of an image of dimensions of $2^N \times 2^N$

- Coef 0 = Image
- Coef 1 = Image
- Coef 2 = Image
- Coef 3 = Image
- Coef 4 = Image
- Coef 5 = Image

![Dot product diagram](dot_product.png)
A Simple, Minimal 2-D Image Transform
The Haar 2-D Wavelet Transform

The 2-D Haar Wavelet Transform corresponds to a modification of this minimal recursive transform.
Invertibility of the 2D Haar Transform

We can recursively reconstruct the intensities of every 2x2 window from its average and detail coefficients.

2 basic operations:
- Sum of 4 pixels
- Difference of pairwise sums of pixels

\[ P_1 = A + d_2 + d_3 + d_4 \]
\[ P_2 = A + d_3 - d_2 - d_4 \]
\[ P_3 = A + d_2 - d_3 - d_4 \]
\[ P_4 = A + d_4 - d_2 - d_3 \]

\[ \frac{1}{4} (P_1 + P_3 - P_2 - P_4) \]
\[ \frac{1}{4} (P_1 + P_2 - P_3 - P_4) \]
\[ \frac{1}{4} (P_1 + P_4 - P_2 - P_3) \]