Topic 03:

Linear Filters & Fourier Analysis

• Filter-based view of image formation
• Linear systems & 1D convolution
• Example 1D filters
• Filtering in 2D
• The Fourier series
• Sampling & aliasing
• Discrete-time filters & the DFT

Today

Covered in detail in the lecture notes
2 ways to think about images

- **pixel-is-a-square (incorrect)**
  - $D(m,n)$
  - Discrete array of pixels

- **pixel-is-a-point-sample (correct)**
  - \( D(x,y) \)
  - Defined over finite domain \([-\pi, \pi] \times [-\pi, \pi]\)

- **pixel-is-a-point-sample (alternative visualization)**

- **continuous image**

- **image after sampling**
Image formation from a filtering perspective

$I(x,y)$

Sensor irradiance

$I(x,y) \forall (x,y) \in [-\pi, \pi] \times [-\pi, \pi]$
Image formation from a filtering perspective

\[ I(x, y) \]

Impulse response \[ f(x, y) \]

Blur due to lens & finite pixel footprint

\[ B(x, y) = (I * f)(x, y) \]

Ideal sensor irradiance

\[ I(x, y) \in [-\pi, \pi] \times [-\pi, \pi] \]
Image formation from a filtering perspective

\[ B(x, y) \]

Blur due to lens & finite pixel footprint

\[ B(x, y) = (I * f)(x, y) \]

Ideal sensor irradiance

\[ I(x, y) \]
Image formation from a filtering perspective

\[ D(x,y) = B(x,y) \cdot S(x,y) \]

\[ S(x,y) = 1 \text{ iff } x, y \text{ are integer multiples of pixel pitch } \Delta x, \Delta y \]
\[ \text{and zero otherwise} \]

\[ B(x,y) = (I * f)(x,y) \]

Blur due to lens & finite pixel footprint

Ideal sensor irradiance

\[ I(x,y)(x,y) \in [-\pi,\pi] \times [-\pi,\pi] \]
Image formation from a filtering perspective

\[ D(x,y) = B(x,y) \cdot S(x,y) \]

\[ S(x,y) = 1 \text{ iff } x, y \text{ integer multiples of } \Delta x, \Delta y \quad \text{zero otherwise} \]

\[ B(x,y) = (I * f)(x,y) \]

Ideal sensor irradiance

\[ I(x,y)(x,y) \in [-\pi, \pi] \times [-\pi, \pi] \]
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Linear filters

- A filter transforms one signal into another.
- Filters are used to describe image formation (lens blur, etc.) as well as to implement operations on images (edge detection, denoising, etc.).
Linear filters

Definition

A transformation $T$ is linear iff it satisfies

$$T[a_1 s_1(x) + a_2 s_2(x)] = a_1 T[s_1(x)] + a_2 T[s_2(x)]$$

for any $a_1, a_2$ and continuous functions $s_1(x), s_2(x)$.
The superposition integral

Every linear transformation can be expressed as an integral of the form

\[ r(x) = \int_{-\infty}^{\infty} f(x, \zeta) s(\zeta) \, d\zeta \]

describes the contribution of the input signal at \( \zeta \) to the output signal at \( x \).
Applying a filter to a delta function

impulse \[ \Rightarrow \]

Filter \[ f(x, \omega) \]

\[ \Rightarrow \]

continuous output signal

Dirac delta function:

\[ \delta(x) = \lim_{\epsilon \to 0} \frac{\text{box}_\epsilon(x)}{\epsilon} \]

Properties of \( \delta(x) \):

1. \( \delta(x) = 0 \quad \forall x \neq 0 \)
2. \( \int f(x) \delta(x) \, dx = f(0) \)

\[ \text{box}_\epsilon(x) = \begin{cases} 1 & |x| < \frac{\epsilon}{2} \\ 0 & |x| \geq \frac{\epsilon}{2} \end{cases} \]
Applying the superposition integral to \( \delta(x) \):

\[
r(x) = \int_{-\infty}^{\infty} f(x, \tau) \delta(\tau) \, d\tau = f(x, 0)
\]
Applying a filter to a delta function

\[ \delta(x-x_0) \]

\[ \Rightarrow \]

\[ \text{Filter} \]

\[ f(x, \tau) \]

\[ \Rightarrow \]

\[ \text{Impulse response} \]

\[ ? \]

Applying the superposition integral to \( \delta(x) \):

\[ r(x) = \int_{-\infty}^{\infty} f(x, \tau) \delta(\tau-x_0) \, d\tau = f(x, x_0) \]
Shift-invariant filtering

A transformation is **shift-invariant** iff shifted impulses produce identical but shifted responses:

\[
r(x) = \int_{-\infty}^{\infty} f(x, \tau) \delta(\tau - x_0) d\tau = \int_{-\infty}^{\infty} f(x, x_0)
\]

= \int_{-\infty}^{\infty} f(x - x_0, 0)
A transformation is **shift-invariant** iff shifted impulses produce identical but shifted responses:

\[
r(x) = \int_{-\infty}^{\infty} f(x, z) \delta(z-x_0) \, dz = f(x, x_0) = f(x-x_0, 0)
\]
Impulse response of a filter

In this case, the transformation is completely described by $f(x) = f(x_0)$ which is called the impulse response.

$$r(x) = \int_{-\infty}^{\infty} f(x, \tau) \delta(\tau - x_0) d\tau = \int f(x, x_0) = f(x - x_0, 0)$$
Impulse response of a filter

In this case, the transformation is completely described by \( f(x) = f(x_0) \) which is called the impulse response.

\[
r(x) = \int_{-\infty}^{\infty} f(x, \tau) \delta(\tau - x_0) \, d\tau = \int_{-\infty}^{\infty} f(x, x_0) = f(x - x_0, 0)
\]
The convolution operation

\[ r(x) = \int_{-\infty}^{\infty} f(x, z) s(z) \, dz = \int_{-\infty}^{\infty} f(x - z) \cdot s(z) \, dz \]
The convolution operation

\[ r(x) = \int_{-\infty}^{\infty} f(x - \tau) \cdot s(\tau) \, d\tau \]

\[ r \overset{\text{def}}{=} f \ast s \]
The convolution operation

\[ r(x) = \int_{-\infty}^{\infty} f(x - \tau) \cdot s(\tau) \, d\tau \]

\[ r \overset{\text{def}}{=} f \ast s \]
Properties of convolution

- Commutativity:
  \[ f \star s = s \star f \]

- Associativity:
  \[ f \star (g \star s) = (f \star g) \star s \]

- Distributivity over addition:
  \[ (f+g) \star s = f \star s + g \star s \]
Signals with a bounded domain

\[ r(x) = \int_{-\infty}^{\infty} f(x) \cdot s(2-x) \, dp \]

So far we assumed that both \( f \) and \( s \) are defined over the real line \((-\infty, \infty)\)
Signals with a bounded domain

\[ r(x) = \int_{-N}^{N} f(x-z) \cdot s(z) \, dz \]

If \( s, f \) are only defined over \([-N, N]\) we must impose boundary conditions.
Signals with a bounded domain

\[ r(x) = \int_{-N}^{N} f(x - \xi) \cdot s(\xi) \, d\xi \]

If \( s, f \) are only defined over \([-N, N]\), we must impose boundary conditions.
Boundary conditions: zero padding

Assume both signal and impulse response are zero outside the interval (but shift invariance is lost)
Boundary conditions: endpoint padding

$$r(x) = \int_{-\infty}^{\infty} f(x - \tau) \cdot s(\tau) \, d\tau$$

Pad the signal with the values of $s$ at the two endpoints (shift invariance lost here too)
Boundary conditions: periodic signal

\[ r(x) = \int_{-\infty}^{\infty} f(x - z) \cdot s(z) \, dz \]

Assume signal is periodic & impulse response is zero outside \([-N, N]\)
Setting $u = x - 2$ we get

$$r(x) = \int_{-\infty}^{\infty} f(u) s(x-u) \, du$$
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Gaussian in 1D

The probability density function of a 1D Gaussian distribution is given by:

$$G_6(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

The diagram illustrates the Gaussian distribution with mean $m$ and standard deviation $\sigma$. The area under the curve is normalized such that the total probability is 1. The distribution is symmetric around the mean, and the spread of the distribution is determined by the standard deviation. The values $m-3\sigma$, $m-\sigma$, $m$, $m+\sigma$, and $m+3\sigma$ mark the points at three standard deviations from the mean.
1D smoothing with Gaussian filters

$s$

$S \ast G_2$

$S \ast G_4$

No smoothing

$\sigma = 2$

$\sigma = 4$
Derivative filters

Gaussian-smoothed signal:
\[ r(x) = \int G_6(x-z) s(z) \, dz \]

Differentiating the result:
\[ \frac{d^n}{dx^n} r(x) = \frac{d^n}{dx^n} \int G_6(x-z) s(z) \, dz \]
\[ = \int \left[ \frac{d^n}{dx^n} G_6(x-z) \right] \cdot s(z) \, dz \]
\[ = \left[ \frac{d^n}{dx^n} G_6 \right] \ast s \]

* can be computed analytically
$1^{st}$ derivative of a Gaussian

$$\frac{d}{dx} \left( \frac{G_{\frac{3}{\sigma}}}{3} (x) \right)$$
Input image
Convolve each row with a Gaussian derivative filter

\[ \text{row} \ast \frac{d}{dx} G_x \]
Convolve each column with a Gaussian derivative filter

\[ \text{col} \ast \frac{d}{dx} G_x \]
2\textsuperscript{nd} derivative of a Gaussian

\[
\frac{d^2}{dx^2} [G_{\sigma}(x)] \text{ for } \sigma = 1.6667 \text{ mean=0}
\]

![Plot of second derivative of a Gaussian function]
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Convolution in 2D

\[ r(x,y) = \int \int f(u,v) s(x-u, y-v) \, du \, dv \]

\[ \Rightarrow \quad f(x,y) \]

\[ \Rightarrow \quad r(x,y) \]

\[ r \quad \text{def} \quad f \ast s \]
A 2D filter is **separable** if

\[ f(x, y) = f_1(x) f_2(y) \]

where \( f_1, f_2 \) are 1D filters
2D convolution as cascade of 1D convolutions

\[ r(x,y) = \iint f_1(u) \cdot f_2(v) \cdot s(x-u, y-v) \, du \, dv \]

\[ \ast = \int f_2(v) \left[ \int f_1(u) \cdot s(x-u, y-v) \, du \right] \, dv \]

1D convolution along \( x \)

1D convolution along \( y \)

A 2D filter is separable if

\[ f(x,y) = f_1(x) \cdot f_2(y) \]

where \( f_1, f_2 \) are 1D filters
Gaussian in 2D

\[ G(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \]

\[ G(x, y) = G_x(x) \cdot G_y(y) \]
Difference-of-Gaussian filters (DOG filters)

\[ I \ast G_{\sigma_1}(x,y) \]

near 0

original
What does smoothing take away?

$I * G_{G_2}(x,y)$
Difference of two Gaussian-smoothed versions of $I$:

$I * G_{6_1} - I * G_{6_2} = I * (G_{6_1} - G_{6_2})$

the DOG filter
Equivalence of DOG & 2\textsuperscript{nd} derivative filter in 1D

1. Consider $G_{\delta_2}\left(x\right)$ to be a function of $\delta$ and calculate its derivative with respect to $\delta$:

$$G_{\delta_2}\left(x\right) = G\left(x; \delta_1\right)$$

$$G_{\delta_2}\left(x\right) = G\left(x; \delta_2\right)$$

(\delta_2 > \delta_1)

$$\frac{\partial G_\delta}{\partial \delta} \text{ scaled by } (\delta_2 - \delta_1)$$

2. Compare to the 2\textsuperscript{nd} derivative of $G$ with respect to $x$:

$$\frac{\partial^2 G}{\partial x^2}\left(x; \delta\right)$$

The DOG filter is just a scaled version of the Gaussian 2\textsuperscript{nd} derivative filter.
Equivalence of DOG & 2\textsuperscript{nd} derivative filter in 1D

1. Consider $G_b(x)$ to be a function of $b$ and calculate its derivative with respect to $b$:

\[ \frac{\partial G_b(x)}{\partial b} = \frac{G(x;b_2) - G(x;b_1)}{b_2 - b_1} \]

Approximate difference by the derivative at scale $b_1$:

\[ \frac{\partial G_b(x)}{\partial b} \approx G(x;b_1) \cdot \frac{\partial G_b(x)}{\partial b}(x;b_1) \]

2. Compare to the 2\textsuperscript{nd} derivative of $G$ with respect to $x$:

\[ \frac{\partial^2 G_b(x)}{\partial x^2} = \left( \frac{x^2}{b_2^2 - 1} \right) \frac{1}{b_2} G_b(x) \]

\[ \frac{\partial G_b(x)}{\partial b} = \left( \frac{x^2}{b_2^2 - 1} \right) \frac{1}{b} G_b(x) \]

i.e.

\[ \frac{\partial G_b(x)}{\partial b} = b \cdot \frac{\partial^2 G_b(x)}{\partial x^2} \]

\[ G_{b_2}(x) - G_{b_1}(x) \approx (b_2 - b_1) \cdot b_1 \cdot \frac{\partial^2 G_b(x)}{\partial x^2} \]
Difference-Of-Gaussians (DOG) filter in 2D

\[ G_{\sigma_2}(x, y) - G_{\sigma_1}(x, y) \approx \sigma_1 (G_{\sigma_1} - G_{\sigma_2}) \left[ \frac{\partial^2}{\partial x^2} G_{\sigma_1} + \frac{\partial^2}{\partial y^2} G_{\sigma_1} \right] \]

Laplacian filter \( \nabla^2 G_{\sigma_1} \)
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The 2D (Continuous-Time) Fourier Transform

We treat images as infinite-size, continuous periodic functions.

Carpet image (black & white)
The 2D (Continuous-Time) Fourier Transform

We can write such functions as an infinite sum of Fourier Basis images, each corresponding to a different spatial frequency.

Carpet image (black & white) = $F_{5,0}$ + $F_{10,0}$ + $F_{0,5}$ + $F_{5,5}$ + ...

The Fourier Basis Images (real component)
The 2D Discrete Fourier Transform

We can write such functions as an infinite sum of Fourier Basis images, each corresponding to a different spatial frequency.
Consider a bounded, continuous, possibly complex periodic signal with $S(x) = S(x + 2\pi)$.

Then we can express it as

$$S(x) = \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + \cdots + a_n \cos(nx) + \cdots + b_1 \sin(x) + b_2 \sin(2x) + \cdots + b_n \sin(nx) + \cdots$$

*contributions of n-th harmonic

periodic over $[-\pi, \pi]$
Today

- A1 due NOW (electronic subs only)
- A2 due in 3 weeks
- Three topics today
  - Wrap up Fourier transforms
  - Edge detection basics
  - Intro to robust estimation
Orthogonality of $\sin(x)$, $\cos(x)$ over $(-\pi, \pi)$

\[
\int_{-\pi}^{\pi} \sin(mx) \cdot \sin(mx) \, dx = \begin{cases} 
0 & \text{if } n \neq m \\
\pi & \text{if } n = m > 0
\end{cases}
\]

\[
s(x) = \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + \ldots + a_n \cos(nx) + \ldots
\]

\[
+ b_1 \sin(x) + b_2 \sin(2x) + \ldots + b_n \sin(nx) + \ldots
\]
Orthogonality of $\sin(x)$, $\cos(x)$ over $(-\pi, \pi)$

Multiplying with $\sin(nx)$ and integrating,

\[
\int_{-\pi}^{\pi} \sin(nx) s(x) \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \sin(nx) + \sum_{m=1}^{\infty} a_m \cos(mx) \sin(nx) + \sum_{m=1}^{\infty} b_m \sin(mx) \sin(nx) \, dx
\]

\[
s(x) = \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + \cdots + a_n \cos(nx) + \cdots + b_1 \sin(x) + b_2 \sin(2x) + \cdots + b_n \sin(nx) + \cdots
\]
The Fourier series coefficients

More generally

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) s(x) \, dx \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) s(x) \, dx \]

\[ \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} s(x) \, dx \quad \text{average value of signal} \]

\[ s(x) = \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + \cdots + a_n \cos(nx) + \cdots \]

\[ + b_1 \sin(x) + b_2 \sin(2x) + \cdots + b_n \sin(nx) + \cdots \]
Using Euler notation

More generally

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) s(x) \, dx \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) s(x) \, dx \]

\[ \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} s(x) \, dx \quad \leftrightarrow \quad \text{average value of signal} \]

**Euler notation:**

\[ e^{j\theta} = \cos\theta + j\sin\theta \]
Fourier Transform of a continuous periodic signal

These relations can now be expressed more compactly:

$S(x) = \sum_{n=-\infty}^{\infty} F_n e^{jnx}$

Fourier Series (aka Inverse Fourier Transform)

$F_n = \int_{-\pi}^{\pi} S(x) e^{-jnx} \, dx$

Euler notation:

$e^{j\theta} = \cos \theta + j \sin \theta$
Fourier Transform of a continuous periodic signal

For periodic $s(x)$ with domain $[0,1]$ and period $1$:

$$s(x) = \sum_{n=-\infty}^{\infty} F_n e^{jn2\pi x}$$

(Continuous-Time) Fourier Transform of $s(x)$

$$F_n = \int_{0}^{1} s(x) e^{-jn2\pi x} \, dx$$

Euler notation:

$$e^{j\theta} = \cos \theta + j \sin \theta$$

Fourier Series (aka Inverse Fourier Transform)
Conventions used here

• Use \( w \) to denote frequency:

\[ w_n = \begin{cases} 2\pi n & \text{if } s(x) \in \mathbb{C}[0, \pi] \\ n & \text{if } s(x) \in \mathbb{C}[-\pi, \pi] \end{cases} \]

• \( F_n \{ s \} \) is complex-valued in general, expressed in the form \( p e^{i\phi} \):

\[
p(w_n) = | F_n \{ s \}| = |F_n| \quad \phi(w_n) = \arg(F_n \{ s \})
\]
Fourier Transform properties

Symmetries:

\[ \forall x \ s(x) \in \mathbb{R} \implies \mathcal{F}_n \{ s(x) \} \text{ is symmetric} \]
\[ \mathcal{F}_n \{ s(x) \} = \mathcal{F}_n^* \quad \text{conjugate} \]

\[ \forall x \ s(x) = s(-x) \implies \mathcal{F}_n \{ s(x) \} \text{ is real-valued} \]

\[ \forall x \ s(x) = -s(-x) \implies \mathcal{F}_n \{ s(x) \} \text{ is imaginary} \]

Shift property:

\[ \mathcal{F}_n \{ s(x-x_0) \} = e^{-j\omega \times x_0} \mathcal{F}_n \{ s(x) \} \]
The Convolution Theorem

\[ \mathcal{F}\{f \ast s\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{s\} \]

The F.T. of the convolution = the product of the F.T.s of the two functions

**Proof**

\[ \mathcal{F}\{f \ast s\} = \int_{-\infty}^{\infty} (f \ast s)(x) e^{-j\omega x} \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) s(x - \tau) e^{-j\omega x} \, d\tau \, dx \]

\[ = \int_{-\infty}^{\infty} f(\tau) \left[ \int_{-\infty}^{\infty} s(x - \tau) e^{-j\omega x} \, dx \right] d\tau \]

\[ = \int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} \cdot \mathcal{F}\{s\}(\omega) \, d\tau = \mathcal{F}\{f\} \cdot \mathcal{F}\{s\} \]
The Convolution Theorem

\[ F \{ f * s \} = F \{ f \} \cdot F \{ s \} \]

the F.T. of the convolution = the product of the F.T.s of two functions

\[ F \{ f \cdot s \} = F \{ f \} \ast F \{ s \} \]

the F.T. of the product = the convolution of the F.T.s of two functions
2D Fourier Transform

For periodic $s(x,y)$ with domain $[-\pi, \pi] \times [-\pi, \pi]$ period 1:

$$s(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{n,m} e^{j(nx+my)}$$

$$F_{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} s(x,y) e^{-j(nx+my)} \, dx dy$$
Example 2D Fourier basis functions

2D Fourier Basis Functions

Grating for $(k,l) = (1,3)$

Grating for $(k,l) = (7,1)$

Real

Imag

Real

$\lambda_l = \frac{2\pi}{\omega_l}$

$\lambda_m = \frac{2\pi}{||\omega||}$

$\lambda_k = \frac{2\pi}{\omega_k}$

zero-crossings of $\sin(\omega_k n + \omega_m m)$

Blocks image and its amplitude spectrum
Examples of 2D filters & their spectra

Common Filters and their Spectra

Top Row: Image of Al and a low-pass (blurred) version of it. The low-pass kernel was separable, composed of 5-tap 1D impulse responses $\frac{1}{16}(1, 4, 6, 4, 1)$ in the $x$ and $y$ directions.

Bottom Row: From left to right are the amplitude spectrum of Al, the amplitude spectrum of the impulse response, and the product of the two amplitude spectra, which is the amplitude spectrum of the blurred version of Al. (Brightness in the left and right images is proportional to log amplitude.)
Examples of 2D filters & their spectra

Common Filters and their Spectra (cont)

From left to right is the original Al, a **high-pass** filtered version of Al, and the amplitude spectrum of the filter. This impulse response is defined by $\delta(n) - h(n, m)$ where $h[n, m]$ is the separable blurring kernel used in the previous figure.

From left to right is the original Al, a **band-pass** filtered version of Al, and the amplitude spectrum of the filter. This impulse response is defined by the difference of two low-pass filters.
Examples of 2D filters & their spectra

Common Filters and their Spectra (cont)

Top Row: Convolution of Al with a horizontal derivative filter, along with the filter’s Fourier spectrum. The 2D separable filter is composed of a vertical smoothing filter (i.e., $\frac{1}{4}(1, 2, 1)$) and a first-order central difference (i.e., $\frac{1}{2}(-1, 0, 1)$) horizontally.

Bottom Row: Convolution of Al with a vertical derivative filter, and the filter’s Fourier spectrum. The filter is composed of a horizontal smoothing filter and a vertical first-order central difference.
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Image formation from a filtering perspective

\[ B(x, y) \]  

Impulse train

Sampled image

\[ D(x, y) = B(x, y) \cdot S(x, y) \]

\[ S(x, y) = 1 \text{ iff } x, y \text{ integer multiples of pixel pitch } \Delta x, \Delta y \]

\[ \text{zero otherwise} \]

Blur due to lens & finite pixel footprint

\[ B(x, y) = (I * f)(x, y) \]

Ideal sensor irradiance

\[ I(x, y) (x, y) \in [-\pi, \pi] \]
Image formation from a filtering perspective

\[ D(x,y) = B(x,y) \cdot S(x,y) \]

\[ S(x,y) = 1 \text{ if } x, y \text{ integer multiples of} \]

\[ \text{pixel pitch } \Delta x, \Delta y \]

\[ 0 \text{ otherwise} \]

\[ B(x,y) = (I * f)(x,y) \]

Ideal sensor irradiance

\[ I(x,y) (x,y) \in \mathbb{R} \]

\[ [-\pi, \pi] \times [-\pi, \pi] \]
Example: photo with no visible aliasing

Close-up photo of my office carpet (looks pretty good)
Aliased photo
Sampling in spatial vs. Fourier domain

Let us consider the sampling operation in the Fourier Transform domain:

\[
\sum_{\kappa=-N}^{N} \sum_{\ell=-M}^{M} \delta(x - \frac{\kappa}{N}) \delta(y - \frac{\ell}{M})
\]

\[\times \quad S(x, y) \quad \overset{\text{2N samples}}{\longrightarrow} \]

\[\overset{\text{2N samples}}{\longrightarrow} \quad B(x, y) \quad \overset{\text{F}\{\text{B}\}}{\longrightarrow} \]

\[\overset{\text{F}\{\text{S}\}}{\longrightarrow} \quad ??? \quad \overset{\ast}{\longrightarrow} \]
Sampling in spatial vs. Fourier domain

Let us consider the sampling operation in the Fourier Transform domain:

\[
\sum_{k=-N}^{N} \sum_{l=-M}^{M} \int_{-\frac{N}{M}}^{\frac{N}{M}} \int_{-\frac{M}{N}}^{\frac{M}{N}} \delta(x - \frac{n}{N}) \delta(y - \frac{m}{M}) e^{-j(xn + ym)} \, dx \, dy
\]

\[
S(x, y) \quad \star \quad F\{S\} \\
B(x, y) \quad \star \quad F\{B\}
\]

2N samples

M samples
Sampling in spatial vs. Fourier domain

Sample distances in spatial domain inversely proportional to Fourier domain

\[
\sum_{k=-N}^{N} \sum_{l=-M}^{M} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(x - \frac{m}{N}) \delta(y - \frac{n}{M}) e^{-j(xn + ym)} dx \, dy
\]

\(2N\) samples

\(\frac{M}{N}\) samples

\(S(x, y)\)

\(F\{S\}\)

\(B(x, y)\)

\(F\{B\}\)
Convolution with an impulse train

a superposition of shifted copies of $F\{B\}$
Sampling in spatial vs. Fourier domain

When region of support of $F\{B\}$ does not overlap with adjacent copies it is possible to reconstruct $B$ from $F^{-1}\{F\{B \ast S\}\}$.
Aliasing

If overlap occurs in $F \times B$, it is not possible to reconstruct $F \times B$ any longer.
Nyquist Sampling Theorem

Let $f(x)$ be a band-limited signal such that $F_w[\xi] = 0$ for $|\xi| > w_0$

Then $f$ uniquely determined by its samples $g(n) = f(n \cdot \Delta x)$ when $\frac{2\pi}{\Delta x} > 2w_0$
The Source of Aliasing…

The end result

• High frequencies are “masked” (aliased) as lower frequencies
Topic 03:

Linear Filters & Fourier Analysis

• Filter-based view of image formation
• Linear systems & 1D convolution
• Example 1D filters
• Filtering in 2D
• The Fourier series
• Sampling & aliasing
• Discrete-time filters & the DFT