On the solution of elliptic partial differential equations on regions with corners II: Detailed analysis

Kirill Serkh

Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, United States

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ABSTRACT

In this paper we investigate the solution of boundary value problems on polygonal domains for elliptic partial differential equations. Previously, we observed that when the boundary value problems are formulated as boundary integral equations of classical potential theory, the solutions are representable by series of elementary functions, to arbitrary order, for all but finitely many values of the angles. Here, we extend this observation to all values of the angles. We show that the solutions near corners are representable, to arbitrary order, by linear combinations of certain non-integer powers and non-integer powers multiplied by logarithms.

1. Introduction

In classical potential theory, elliptic partial differential equations are reduced to second kind boundary integral equations by representing the solutions to the differential equations by single-layer or double-layer potentials on the boundaries of the regions. When the boundaries have corners, the solutions to both the differential and integral equations are singular at the corners.

The properties of the solutions to both the differential and integral equations on regions with corners have been the subject of extensive research (see, for example, [14,21,20]; reviews of the literature can be found in, for example, [10,15]), and the existence and uniqueness of the solutions in the $L^2$-sense have long been known (see [7,19]). While much is known about the spaces to which solutions belong, and the leading singular terms of the solutions are well-known, it was recently observed in [17] that there exists an explicit representation for the solutions which appears to have been overlooked.

E-mail address: kserkh@cims.nyu.edu.

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In [17], we show that when the second kind boundary integral equations of potential theory are solved on polygonal domains, the solutions near corners are explicitly representable to arbitrary order by linear combinations of certain non-integer powers, except at finitely many values of the angles. In this paper, we extend the results of [17] to all values of the angles. In particular, we show that the solutions to the boundary integral equations near corners are, when the boundary data is smooth on each side of the corner, representable to arbitrary order and for any angle by linear combinations of certain non-integer powers and non-integer powers multiplied by logarithms. In proving this, we construct a mapping from the coefficients of these singular terms to the coefficients of the Taylor series representing the boundary data, and show that this mapping is invertible for any value of the angle.

The structure of the paper is as follows. Section 2 provides an overview of the principal results of the paper. In Section 3, we introduce the necessary mathematical preliminaries. Section 4 contains the analytical apparatus. Sections 5 and 6 analyze the Neumann and Dirichlet cases respectively.

Remark 1.1. The numerical solution of the boundary integral equations of potential theory on regions with corners has been the subject of much research, and in recent years significant progress has been made (see, for example, [11,12,9,3,4] and [5]).

2. Overview

The following two subsections 2.1 and 2.2 summarize the Neumann and Dirichlet cases respectively. The principal results of this paper are Theorems 2.1 and 2.2.

Suppose that $\gamma: [-1, 1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with a corner at $\gamma(0)$, and with interior angle $\pi \alpha$. Suppose further that $\gamma$ is parameterized by arc length, and let $\nu(t)$ denote the inward-facing unit normal to the curve $\gamma$ at $t$. Let $\Gamma$ denote the set $\gamma([-1, 1])$. By extending the sides of the wedge to infinity, we divide $\mathbb{R}^2$ into two open sets $\Omega_1$ and $\Omega_2$ (see Fig. 1).

2.1. The Neumann Case

Let $\phi: \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ denote the potential induced by a charge distribution on $\gamma$ with density $\rho: [-1, 1] \to \mathbb{R}$. In other words, let $\phi$ be defined by the formula

$$\phi(x) = -\frac{1}{2\pi} \int_{-1}^{1} \log(||\gamma(t) - x||)\rho(t)\,dt,$$

for all $x \in \mathbb{R}^2 \setminus \Gamma$, where $|| \cdot ||$ denotes the Euclidean norm. Suppose that $g: [-1, 1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \lim_{x \to \gamma(t)} \frac{\partial \phi(x)}{\partial \nu(t)},$$

Fig. 1. A wedge in $\mathbb{R}^2$. Please cite this article in press as: K. Serkh, On the solution of elliptic partial differential equations on regions with corners II: Detailed analysis, Appl. Comput. Harmon. Anal. (2017), http://dx.doi.org/10.1016/j.acha.2017.05.001
for all \(-1 \leq t \leq 1\), i.e. \(g\) is the limit of the normal derivative of integral (1) when \(x\) approaches the point \(\gamma(t)\) from outside. It is well known that

\[
g(s) = \frac{1}{2} \rho(s) + \frac{1}{2\pi} \int_{-1}^{1} K(s, t) \rho(t) \, dt, \tag{3}
\]

for all \(-1 \leq s \leq 1\), where

\[
K(s, t) = \frac{\langle \gamma(t) - \gamma(s), \nu(s) \rangle}{\|\gamma(t) - \gamma(s)\|^2}, \tag{4}
\]

for all \(-1 \leq s, t \leq 1\), where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(\mathbb{R}^2\).

In this paper, we prove the following theorem, which holds for any angle \(0 < \pi \alpha < 2\pi\), and which is the first of the two principal results of this paper.

**Theorem 2.1.** Suppose that \(0 < \alpha < 2\) and that \(N\) is a positive integer. Letting \([\cdot]\) and \(\lfloor \cdot \rfloor\) denote the floor and ceiling functions respectively, suppose that

\[
\mathcal{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{5}
\]

\[
\mathcal{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{6}
\]

and

\[
\mathcal{M} = \left\lfloor \frac{(2 - \alpha)N}{2} \right\rfloor, \tag{7}
\]

\[
\mathcal{M} = \left\lceil \frac{(2 - \alpha)N}{2} \right\rceil, \tag{8}
\]

and observe that \(\mathcal{L} + \mathcal{M} = N\) and \(\mathcal{M} + \mathcal{L} = N\). Suppose further that \(\rho\) is defined by the formula

\[
\rho(t) = \sum_{i=1}^{\mathcal{L}} b_i |t|^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\mathcal{M}} b_{\mathcal{L}+i} |t|^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\mathcal{M}} b_{\mathcal{L}+i} |t|^{\frac{2i-1}{\alpha}-1} \log(|t|) + \sum_{i=1}^{\mathcal{M}} c_i \text{sgn}(t) |t|^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\mathcal{M}} c_{\mathcal{M}+i} \text{sgn}(t) |t|^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\mathcal{M}} c_{\mathcal{M}+i} \text{sgn}(t) |t|^{\frac{2i-1}{\alpha}-1} \log(|t|), \tag{9}
\]

for all \(-1 \leq t \leq 1\), where \(b_1, b_2, \ldots, b_N\) and \(c_1, c_2, \ldots, c_N\) are arbitrary real numbers,

\[
S = \left\{ i \in \mathbb{Z}, 1 \leq i \leq \mathcal{M} : \frac{2i}{\alpha} = \frac{2j-1}{\alpha} \text{ for some integer } 1 \leq j \leq \mathcal{L} \right\}, \tag{10}
\]

and

\[
T = \left\{ i \in \mathbb{Z}, 1 \leq i \leq \mathcal{L} : \frac{2i}{\alpha} = \frac{2j-1}{\alpha} \text{ for some integer } 1 \leq j \leq \mathcal{M} \right\}. \tag{11}
\]

Suppose finally that \(g\) is defined by (3). Then there exist sequences of real numbers \(\beta_0, \beta_1, \ldots\) and \(\gamma_0, \gamma_1, \ldots\) such that

Fig. 2. The functions $\frac{2n}{m} - 1$, for positive integers $n$, are plotted as dashed lines, while the functions $\frac{2m-1}{n} - 1$, for positive integers $m$, are plotted as solid lines. The values of $\alpha$ at which the two graphs intersect correspond to the appearance of log terms in (9).

\[ g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \text{sgn}(t)|t|^n, \]  

for all $-1 \leq t \leq 1$. Conversely, suppose that $g$ has the form (12). Suppose further that $N$ is an arbitrary positive integer. Then, for all angles $\pi\alpha$, there exist unique real numbers $b_1, b_2, \ldots, b_N$ and $c_1, c_2, \ldots, c_N$ such that $\rho$, defined by (9), solves equation (3) to within an error $O(t^N)$.

In other words, if $\rho$ has the form (9), then $g$ is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$. Conversely, if $g$ is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$, then for each positive integer $N$, and for each angle $\pi\alpha$, there exists a unique solution $\rho$ of the form (9) that solves equation (3) to within an error $O(t^N)$.

**Remark 2.1.** It is easy to observe that when $\alpha$ is irrational, the sets $S$ and $T$, defined by (10) and (11) respectively, are empty for all $N$. In Section 3.5, we show, more precisely, that there exists some positive integer $N_0$ such that $S$ is not empty for all integers $N \geq N_0$ if and only if

\[ \alpha = \frac{4m - 2}{2n + 2m - 1}, \]  

for some positive integers $n$ and $m$; likewise, we show that there exists some positive integer $N_0$ such that $T$ is not empty for all integers $N \geq N_0$ if and only if

\[ \alpha = \frac{4n}{2n + 2m - 1}, \]  

for some positive integers $n$ and $m$ (see Theorems 3.15 and 3.16). Fig. 2 illustrates the values of $\alpha$ for which the set $S$ is nonempty.

**Observation 2.2.** The series (9) is related to the solutions to the Cauchy–Euler equation (see, for example, [2]) in the following way.

Suppose that $0 < \nu < 2$, and that $\frac{2n}{2-\nu} = \frac{2m-1}{\nu}$ for some positive integers $n$ and $m$. We observe that, when $\alpha \neq \nu$, (9) contains a linear combination of the pair of functions

\[ |t|^{\frac{2m-1}{\nu} - 1}, |t|^{\frac{2n}{\nu} - 1}, \]  

which are linearly independent. However, when \( \alpha = \nu \), this pair of functions is no longer linearly independent, and is replaced in (9) by a linear combination of

\[
|t|^{\frac{2m-1}{\alpha}} - |t|^{\frac{2n-1}{\nu}} \log(|t|),
\]

which are linearly independent. The same kind of observation can be made about the odd terms in (9).

This is similar to the behavior of the solutions to the Cauchy–Euler equation

\[
r^2 \frac{d^2 y}{dr^2} + ar \frac{dy}{dr} + by = 0,
\]

where \( 0 < r \leq 1 \) and \( a \) and \( b \) are real numbers, which arises when Laplace’s equation is solved in polar coordinates (see, for example, [2]). Suppose now that \( \mu_1, \mu_2 \) are the solutions to the quadratic equation

\[
\mu^2 + (a - 1)\mu + b = 0.
\]

When \( \mu_1 \neq \mu_2 \), the solution to (17) is a linear combination of the pair of functions

\[
r^{\mu_1}, r^{\mu_2},
\]

which are linearly independent. However, when \( \mu_1 = \mu_2 \), this pair of functions is no longer linearly independent, and the solution to (17) is a linear combination of

\[
r^{\mu_1}, r^{\mu_2} \log(r),
\]

which are linearly independent.

**Observation 2.3.** The representation of the solution to (1) by the series (9) calls to mind the method of Frobenius, in which the solutions to a second-order linear ordinary differential equation are represented by infinite series of possibly non-integer powers, called Frobenius series (see, for example, [6]).

### 2.2. The Dirichlet case

Let \( \phi: \mathbb{R}^2 \setminus \Gamma \to \mathbb{R} \) denote the potential induced by a dipole distribution on \( \gamma \) with density \( \rho: [-1, 1] \to \mathbb{R} \). In other words, let \( \phi \) be defined by the formula

\[
\phi(x) = \frac{1}{2\pi} \int_{-1}^{1} \frac{\langle x - \gamma(t), \nu(t) \rangle}{\|x - \gamma(t)\|^2} \rho(t) \, dt,
\]

for all \( x \in \mathbb{R}^2 \setminus \Gamma \), where \( \| \cdot \| \) denotes the Euclidean norm and \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^2 \). Suppose that \( g: [-1, 1] \to \mathbb{R} \) is defined by the formula

\[
g(t) = \lim_{x \to \gamma(t)} \phi(x),
\]

for all \(-1 \leq t \leq 1\), i.e. \( g \) is the limit of integral (21) when \( x \) approaches the point \( \gamma(t) \) from inside. It is well known that

\[
g(s) = \frac{1}{2} \rho(s) + \frac{1}{2\pi} \int_{-1}^{1} K(t, s) \rho(t) \, dt,
\]
for all $-1 \leq s \leq 1$, where
\[
K(t, s) = \frac{\langle \gamma(s) - \gamma(t), \nu(t) \rangle}{\|\gamma(s) - \gamma(t)\|^2},
\] (24)
for all $-1 \leq s, t \leq 1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$.

In this paper, we prove the following theorem, which holds for any angle $0 < \pi \alpha < 2\pi$, and which is the second of the two principal results of this paper.

**Theorem 2.2.** Suppose that $0 < \alpha < 2$ and that $N$ is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that
\[
\mathcal{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{25}
\]
and
\[
\mathcal{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{26}
\]
and observe that $\mathcal{L} + \mathcal{M} = N$ and $\mathcal{M} + \mathcal{L} = N$. Suppose further that $\rho$ is defined by the formula
\[
\rho(t) = b_0 + \sum_{i=1}^{\mathcal{T}} b_i |t|^{\frac{2i-1}{\alpha}} + \sum_{i \in \mathcal{S}} b_{\mathcal{L}+i} |t|^{\frac{2i}{\alpha}} + \sum_{i=1}^{\mathcal{M}} b_{\mathcal{L}+i} |t|^{\frac{2i}{\alpha}} \log(|t|) + c_0 \cdot \text{sgn}(t)
\]
\[+
\sum_{i=1}^{\mathcal{M}} c_i \text{sgn}(t) |t|^{\frac{2i-1}{\alpha}} + \sum_{i=1}^{\mathcal{L}} c_{\mathcal{M}+i} \text{sgn}(t) |t|^{\frac{2i}{\alpha}} + \sum_{i=1}^{\mathcal{L}} c_{\mathcal{M}+i} \text{sgn}(t) |t|^{\frac{2i}{\alpha}} \log(|t|), \tag{29}
\]
for all $-1 \leq t \leq 1$, where $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ are arbitrary real numbers,
\[
S = \left\{ i \in \mathbb{Z}, 1 \leq i \leq \mathcal{M} : \frac{2i}{\alpha} = \frac{2j - 1}{\alpha} \text{ for some integer } 1 \leq j \leq \mathcal{T} \right\}, \tag{30}
\]
and
\[
T = \left\{ i \in \mathbb{Z}, 1 \leq i \leq \mathcal{L} : \frac{2i}{\alpha} = \frac{2j - 1}{\alpha} \text{ for some integer } 1 \leq j \leq \mathcal{M} \right\}. \tag{31}
\]
Suppose finally that $g$ is defined by (23). Then there exist sequences of real numbers $\beta_0, \beta_1, \ldots$ and $\gamma_0, \gamma_1, \ldots$ such that
\[
g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \text{sgn}(t) |t|^n, \tag{32}
\]
for all $-1 \leq t \leq 1$. Conversely, suppose that $g$ has the form (32). Suppose further that $N$ is an arbitrary positive integer. Then, for all angles $\pi \alpha$, there exist unique real numbers $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ such that $\rho$, defined by (29), solves equation (23) to within an error $O(t^{N+1})$. 

In other words, if \( \rho \) has the form (29), then \( g \) is smooth on each of the intervals \([-1,0]\) and \([0,1]\). Conversely, if \( g \) is smooth on each of the intervals \([-1,0]\) and \([0,1]\), then for each positive integer \( N \), and for each angle \( \pi \alpha \), there exists a unique solution \( \rho \) of the form (29) that solves equation (23) to within an error \( O(t^{N+1}) \).

3. Mathematical preliminaries

3.1. Boundary value problems

Suppose that \( \gamma: [0,L] \rightarrow \mathbb{R}^2 \) is a simple closed curve of length \( L \) with a finite number of corners (see Fig. 3). Suppose further that \( \gamma \) is analytic except at the corners. We denote the interior of \( \gamma \) by \( \Omega \) and the boundary of \( \Omega \) by \( \Gamma \), and let \( \nu(t) \) denote the normalized internal normal to \( \gamma \) at \( t \in [0,L] \). Supposing that \( g \) is some function \( g: [0,L] \rightarrow \mathbb{R} \), we will solve the following four problems.

1) Interior Neumann problem (INP): Find a function \( \phi: \Omega \rightarrow \mathbb{R} \) such that

\[
\nabla^2 \phi(x) = 0 \quad \text{for} \; x \in \Omega, \quad (33)
\]

\[
\lim_{x \to \gamma(t) \atop x \in \Omega} \frac{\partial \phi(x)}{\partial \nu(t)} = g(t) \quad \text{for} \; t \in [0,L]. \quad (34)
\]

2) Exterior Neumann problem (ENP): Find a function \( \phi: \mathbb{R}^2 \setminus \overline{\Omega} \rightarrow \mathbb{R} \) such that

\[
\nabla^2 \phi(x) = 0 \quad \text{for} \; x \in \mathbb{R}^2 \setminus \overline{\Omega}, \quad (35)
\]

\[
\lim_{x \to \gamma(t) \atop x \in \mathbb{R}^2 \setminus \overline{\Omega}} \frac{\partial \phi(x)}{\partial \nu(t)} = g(t) \quad \text{for} \; t \in [0,L]. \quad (36)
\]

3) Interior Dirichlet problem (IDP): Find a function \( \phi: \Omega \rightarrow \mathbb{R} \) such that

\[
\nabla^2 \phi(x) = 0 \quad \text{for} \; x \in \Omega, \quad (37)
\]

\[
\lim_{x \to \gamma(t) \atop x \in \Omega} \phi(x) = g(t) \quad \text{for} \; t \in [0,L]. \quad (38)
\]

4) Exterior Dirichlet problem (EDP): Find a function \( \phi: \mathbb{R}^2 \setminus \overline{\Omega} \rightarrow \mathbb{R} \) such that

\[
\nabla^2 \phi(x) = 0 \quad \text{for} \; x \in \mathbb{R}^2 \setminus \overline{\Omega}, \quad (39)
\]

\[
\lim_{x \to \gamma(t) \atop x \in \mathbb{R}^2 \setminus \overline{\Omega}} \phi(x) = g(t) \quad \text{for} \; t \in [0,L]. \quad (40)
\]
Suppose that \( g \in L^2([0, L]) \). Then the interior and exterior Dirichlet problems have unique solutions. If \( g \) also satisfies the condition
\[
\int_{0}^{L} g(t) \, dt = 0, \tag{41}
\]
then the interior and exterior Neumann problems have unique solutions up to an additive constant (see, for example, [13,8]).

3.2. Integral equations of potential theory

In classical potential theory, boundary value problems are solved by representing the function \( \phi \) by integrals of potentials over the boundary. The potential of a unit charge located at \( x_0 \in \mathbb{R}^2 \) is the function \( \psi_{x_0}^0: \mathbb{R}^2 \setminus x_0 \to \mathbb{R} \), defined via the formula
\[
\psi_{x_0}^0(x) = \log(\|x - x_0\|), \tag{42}
\]
failing all \( x \in \mathbb{R}^2 \setminus x_0 \), where \( \| \cdot \| \) denotes the Euclidean norm. The potential of a unit dipole located at \( x_0 \in \mathbb{R}^2 \) and oriented in direction \( h \in \mathbb{R}^2 \), \( \|h\| = 1 \), is the function \( \psi_{x_0,h}^1: \mathbb{R}^2 \setminus x_0 \to \mathbb{R} \), defined via the formula
\[
\psi_{x_0,h}^1(x) = \frac{\langle h, x_0 - x \rangle}{\|x_0 - x\|^2}, \tag{43}
\]
failing all \( x \in \mathbb{R}^2 \setminus x_0 \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product.

Charge and dipole distributions with density \( \rho: [0, L] \to \mathbb{R} \) on \( \Gamma \) produce potentials given by the formulas
\[
\phi(x) = \int_{0}^{L} \psi_{t(x)}^0(x) \rho(t) \, dt, \tag{44}
\]
and
\[
\phi(x) = \int_{0}^{L} \psi_{\gamma(t),\nu(t)}^1(x) \rho(t) \, dt, \tag{45}
\]
respectively, for any \( x \in \mathbb{R}^2 \setminus \Gamma \).

3.2.1. Reduction of boundary value problems to integral equations

The following four theorems reduce the boundary value problems of Section 3.1 to boundary integral equations. They are found in, for example, [16], [19].

Theorem 3.1 (Exterior Neumann problem). Suppose that \( \rho \in L^2([0, L]) \). Suppose further that \( g: [0, L] \to \mathbb{R} \) is defined by the formula
\[
g(s) = -\pi \rho(s) + \int_{0}^{L} \psi_{\gamma(s),\nu(s)}^1(\gamma(t)) \rho(t) \, dt, \tag{46}
\]
for any \( s \in [0, L] \). Then \( g \) is in \( L^2([0, L]) \), and a solution \( \phi \) to the exterior Neumann problem with right hand side \( g \) is obtained by substituting \( \rho \) into (44). Moreover, for any \( g \in L^2([0, L]) \), equation (46) has a unique solution \( \rho \in L^2([0, L]) \).
Theorem 3.2 (Interior Dirichlet problem). Suppose that $\rho \in L^2([0, L])$. Suppose further that $g: [0, L] \to \mathbb{R}$ is defined by the formula

$$g(s) = -\pi \rho(s) + \int_0^L \psi^1_{\gamma(t), \nu(t)}(\gamma(s))\rho(t) \, dt,$$  \hspace{1cm} (47)

for any $s \in [0, L]$. Then $g$ is in $L^2([0, L])$, and the solution $\phi$ to the interior Dirichlet problem with right hand side $g$ is obtained by substituting $\rho$ into (45). Moreover, for any $g \in L^2([0, L])$, equation (47) has a unique solution $\rho \in L^2([0, L])$.

The following two theorems make use of the function $\varpi: [0, L] \to \mathbb{R}$, defined as the solution to the equation

$$\int_0^L \varpi(t) \log(||x - \gamma(t)||) \, dt = 1,$$ \hspace{1cm} (48)

for all $x \in \overline{\Omega}$. In other words, we define the function $\varpi$ as the density of the charge distribution on $\Gamma$ when $\overline{\Omega}$ is a conductor.

Theorem 3.3 (Interior Neumann problem). Suppose that $\rho \in L^2([0, L])$. Suppose further that $g: [0, L] \to \mathbb{R}$ is defined by the formula

$$g(s) = \pi \rho(s) + \int_0^L \psi^1_{\gamma(s), \nu(s)}(\gamma(t))\rho(t) \, dt,$$ \hspace{1cm} (49)

for any $s \in [0, L]$. Then $g$ is in $L^2([0, L])$, and a solution $\phi$ to the exterior Neumann problem with right hand side $g$ is obtained by substituting $\rho$ into (44).

Now suppose that $g$ is an arbitrary function in $L^2([0, L])$ such that

$$\int_0^L g(t) \, dt = 0.$$ \hspace{1cm} (50)

Then equation (49) has a solution $\rho \in L^2([0, L])$. Moreover, if $\rho_1$ and $\rho_2$ are both solutions to equation (49), then there exists a real number $C$ such that

$$\rho_1(t) - \rho_2(t) = C \varpi(t),$$ \hspace{1cm} (51)

for $t \in [0, L]$, where $\varpi$ is the solution to (48).

Theorem 3.4 (Exterior Dirichlet problem). Suppose that $\rho \in L^2([0, L])$. Suppose further that $g: [0, L] \to \mathbb{R}$ is defined by the formula

$$g(s) = \pi \rho(s) + \int_0^L \psi^1_{\gamma(t), \nu(t)}(\gamma(s))\rho(t) \, dt,$$ \hspace{1cm} (52)

for any $s \in [0, L]$. Then $g$ is in $L^2([0, L])$, and the solution $\phi$ to the interior Dirichlet problem with right hand side $g$ is obtained by substituting $\rho$ into (45).
Now suppose that \( g \) is an arbitrary function in \( L^2([0,L]) \) such that
\[
\int_0^L g(t)\varpi(t) \, dt = 0,
\] (53)

where \( \varpi \) is the solution to (48). Then equation (52) has a solution \( \rho \in L^2([0,L]) \). Moreover, if \( \rho_1 \) and \( \rho_2 \) are both solutions to equation (52), then there exists a real number \( C \) such that
\[
\rho_1(t) - \rho_2(t) = C,
\] (54)

for \( t \in [0,L] \).

**Observation 3.1.** Equation (46) is the adjoint of equation (47), and equation (49) is the adjoint of equation (52).

**Observation 3.2.** Suppose that the curve \( \gamma: [0,L] \to \mathbb{R}^2 \) is not closed. We observe that if \( \rho \in L^2([0,L]) \), and \( g \) is defined by either (46), (47), (49), or (52), then \( g \in L^2([0,L]) \). Moreover, if \( g \in L^2([0,L]) \), then equations (46), (47), (49), and (52) have unique solutions \( \rho \in L^2([0,L]) \).

### 3.2.2. Properties of the Kernels of Equations (46), (47), (49), and (52)

The following theorem shows that if a curve \( \gamma \) is has \( k \) continuous derivatives, where \( k \geq 2 \), then the kernels of equations (46), (47), (49), and (52) have \( k-2 \) continuous derivatives. It is found in, for example, [1].

**Theorem 3.5.** Suppose that \( \gamma: [0,L] \to \mathbb{R}^2 \) is a curve in \( \mathbb{R}^2 \) that is parameterized by arc length. Suppose further that \( k \geq 2 \) is an integer. If \( \gamma \) is \( C^k \) on a neighborhood of a point \( s \), where \( 0 < s < L \), then
\[
\psi_{\gamma(s),\nu(s)}^1(\gamma(t)) = \psi_{\gamma(t),\nu(t)}^1(\gamma(s)),
\] (55)
\[
\psi_{\gamma(s),\nu(s)}^1(\gamma(t)) = -\frac{1}{2}k(s),
\] (56)
are \( C^{k-2} \) functions of \( t \) on a neighborhood of \( s \) and
\[
\lim_{t \to s} \psi_{\gamma(s),\nu(s)}^1(\gamma(t)) = \lim_{t \to s} \psi_{\gamma(t),\nu(t)}^1(\gamma(s)) = -\frac{1}{2}k(s),
\] (57)
where \( k: [0, L] \to \mathbb{R} \) is the signed curvature of \( \gamma \). Furthermore, if \( \gamma \) is analytic on a neighborhood of a point \( s \), where \( 0 < s < L \), then (55) and (56) are analytic functions of \( t \) on a neighborhood of \( s \).

When the curve \( \gamma \) is a wedge, the kernels of equations (46), (47), (49), and (52) have a particularly simple form, which is given by the following lemma. It is proved in [18].

**Lemma 3.6.** Suppose \( \gamma: [-1,1] \to \mathbb{R}^2 \) is defined by the formula
\[
\gamma(t) = \begin{cases} 
-t \cdot (\cos(\pi \alpha), \sin(\pi \alpha)) & \text{if } -1 \leq t < 0, \\
(t, 0) & \text{if } 0 \leq t \leq 1,
\end{cases}
\] (58)
shown in Fig. 4. Then, for all \( 0 < s \leq 1 \),
\[
\psi_{\gamma(s),\nu(s)}^1(\gamma(t)) = \begin{cases} 
t \sin(\pi \alpha) & \text{if } -1 \leq t < 0, \\
\frac{t \sin(\pi \alpha)}{s^2 + t^2 + 2st \cos(\pi \alpha)} & \text{if } 0 \leq t \leq 1,
\end{cases}
\] (59)
and, for all $-1 \leq s < 0$,

$$\psi_{\gamma(s),\nu(s)}^1(\gamma(t)) = \begin{cases} 0 & \text{if } -1 \leq t < 0, \\ -t \sin(\pi \alpha) & \frac{s^2}{s^2 + t^2 + 2st \cos(\pi \alpha)} & \text{if } 0 \leq t \leq 1. \end{cases}$$  \hspace{1cm} (60)$$

**Corollary 3.7.** Identities (59) and (60) remain valid after any rotation or translation of the curve $\gamma$ in $\mathbb{R}^2$.

**Corollary 3.8.** When the curve $\gamma$ is a straight line, $\psi_{\gamma(s),\nu(s)}^1(\gamma(t)) = 0$ for all $-1 \leq s, t \leq 1$.

### 3.3. Boundary integral equations on a wedge

Suppose that the curve $\gamma: [-1, 1] \to \mathbb{R}^2$ is a wedge defined by (58) with interior angle $\pi \alpha$, where $0 < \alpha < 2$ (see Fig. 4), and let $\nu(t)$ denote the inward-facing unit normal to the curve $\gamma$ at the point $\gamma(t)$.

#### 3.3.1. The Neumann Case

Let $g$ be a function in $L^2([-1, 1])$, and suppose that $\rho \in L^2([-1, 1])$ solves the equation

$$g(s) = -\pi \rho(s) + \int_{-1}^{1} \psi_{\gamma(s),\nu(s)}^1(\gamma(t)) \rho(t) \, dt,$$  \hspace{1cm} (61)$$

for all $s \in [-1, 1]$, where $\psi_{\gamma(s),\nu(s)}^1$ is defined by (43).

The following lemma combines equation (61) with identities (59) and (60).

**Lemma 3.9.** Let $g \in L^2([-1, 1])$, and suppose that $\rho \in L^2([-1, 1])$ solves equation (61). Then

$$g(s) = -\pi \rho(s) - \int_{0}^{1} \frac{t \sin(\pi \alpha)}{s^2 + t^2 + 2st \cos(\pi \alpha)} \rho(t) \, dt,$$  \hspace{1cm} (62)$$

for all $-1 \leq s < 0$, and

$$g(s) = -\pi \rho(s) + \int_{-1}^{0} \frac{t \sin(\pi \alpha)}{s^2 + t^2 + 2st \cos(\pi \alpha)} \rho(t) \, dt,$$  \hspace{1cm} (63)$$

for all $0 < s \leq 1$.

The following theorem uses a symmetry argument to reduce (62), (63) from two coupled integral equations on the interval $[-1, 1]$ to two independent integral equations on the interval $[0, 1]$. It is proved in [17].
Theorem 3.10. Suppose that $g$ is a function in $L^2([-1,1])$, and $\rho \in L^2([-1,1])$ solves equation (61). Suppose further that even functions $g_{\text{even}}, \rho_{\text{even}} \in L^2([-1,1])$ are defined via the formulas

\begin{align*}
g_{\text{even}}(s) = \frac{1}{2} (g(s) + g(-s)), \\
\rho_{\text{even}}(s) = \frac{1}{2} (\rho(s) + \rho(-s)).
\end{align*}

Then

\begin{equation}
g_{\text{even}}(s) = -\pi \rho_{\text{even}}(s) - \int_{0}^{1} \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho_{\text{even}}(t) \, dt,
\end{equation}

for all $0 < s \leq 1$.

Likewise, suppose that odd functions $g_{\text{odd}}, \rho_{\text{odd}} \in L^2([-1,1])$ are defined via the formulas

\begin{align*}
g_{\text{odd}}(s) = \frac{1}{2} (g(s) - g(-s)), \\
\rho_{\text{odd}}(s) = \frac{1}{2} (\rho(s) - \rho(-s)).
\end{align*}

Then

\begin{equation}
g_{\text{odd}}(s) = -\pi \rho_{\text{odd}}(s) + \int_{0}^{1} \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho_{\text{odd}}(t) \, dt,
\end{equation}

for all $0 < s \leq 1$.

3.3.2. The Dirichlet case

Let $g$ be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

\begin{equation}
g(s) = -\pi \rho(s) + \int_{-1}^{1} \psi_{1,\nu(t)}^{(\gamma(t))} \rho(t) \, dt,
\end{equation}

for all $s \in [-1,1]$, where $\psi_{\gamma(t),\nu(t)}$ is defined by (43).

The following lemma combines equation (70) with identities (59) and (60).

Lemma 3.11. Let $g \in L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves equation (70). Then

\begin{align*}
g(s) = -\pi \rho(s) - \int_{0}^{1} \frac{s \sin(\pi \alpha)}{s^2 + t^2 + 2st \cos(\pi \alpha)} \rho(t) \, dt, \\
g(s) = -\pi \rho(s) + \int_{-1}^{0} \frac{s \sin(\pi \alpha)}{s^2 + t^2 + 2st \cos(\pi \alpha)} \rho(t) \, dt,
\end{align*}

for all $-1 \leq s < 0$, and

for all $0 < s \leq 1$.  

The following theorem uses a symmetry argument to reduce (71), (72) from two coupled integral equations on the interval \([-1, 1]\) to two independent integral equations on the interval \([0, 1]\). It is proved in [17].

**Theorem 3.12.** Suppose that \(g\) is a function in \(L^2([-1, 1])\), and \(\rho \in L^2([-1, 1])\) solves equation (70). Suppose further that even functions \(g_{\text{even}}, \rho_{\text{even}} \in L^2([-1, 1])\) are defined via the formulas

\[
g_{\text{even}}(s) = \frac{1}{2}(g(s) + g(-s)),
\]
\[
\rho_{\text{even}}(s) = \frac{1}{2}(\rho(s) + \rho(-s)).
\]

Then

\[
g_{\text{even}}(s) = -\pi \rho_{\text{even}}(s) - \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho_{\text{even}}(t) \, dt,
\]

for all \(0 < s \leq 1\).

Likewise, suppose that odd functions \(g_{\text{odd}}, \rho_{\text{odd}} \in L^2([-1, 1])\) are defined via the formulas

\[
g_{\text{odd}}(s) = \frac{1}{2}(g(s) - g(-s)),
\]
\[
\rho_{\text{odd}}(s) = \frac{1}{2}(\rho(s) - \rho(-s)).
\]

Then

\[
g_{\text{odd}}(s) = -\pi \rho_{\text{odd}}(s) + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho_{\text{odd}}(t) \, dt,
\]

for all \(0 < s \leq 1\).

### 3.4. Chebyshev systems

The following defines the concept of a Chebyshev system.

**Definition 3.1 (Chebyshev system).** Let \(a < b\) be real numbers and let \(n\) be a positive integer. A set of \(n\) functions \(\varphi_1, \varphi_2, \ldots, \varphi_n : [a, b] \to \mathbb{R}\) will be referred to as a Chebyshev system if each function is continuous on \([a, b]\) and the determinants

\[
\det \begin{pmatrix}
\varphi_1(x_1) & \varphi_2(x_1) & \ldots & \varphi_n(x_1) \\
\varphi_1(x_2) & \varphi_2(x_2) & \ldots & \varphi_n(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1(x_n) & \varphi_2(x_n) & \ldots & \varphi_n(x_n)
\end{pmatrix}
\]

are nonzero for any set of \(n\) distinct points \(x_1, x_2, \ldots, x_n \in [a, b]\).

The following theorem states that the Lagrange interpolating polynomials (also called Hermite interpolating polynomials) form a Chebyshev system. For completeness, a proof is provided in Appendix A.
Theorem 3.13. Suppose that $K$ and $L$ are nonnegative integers such that $K \leq L$. Suppose further that $\mu_1, \mu_2, \ldots, \mu_L$ are distinct real numbers. Suppose finally that $p: \mathbb{R} \to \mathbb{R}$ is defined by

$$p(x) = \prod_{\ell=1}^{L} (x - \mu_{\ell}) \prod_{k=1}^{K} (x - \mu_{k}),$$

(80)

for all $x \in \mathbb{R}$. Then the set of $L + K$ functions

$$\left\{ \frac{p(x)}{x - \mu_{\ell}} \right\}_{\ell=1}^{L} \cup \left\{ \frac{p(x)}{(x - \mu_{k})^2} \right\}_{k=1}^{K},$$

(81)

where $x \in \mathbb{R}$, is a Chebyshev system.

3.5. Miscellaneous analytical facts

This section contains a number of miscellaneous elementary technical lemmas.

Theorem 3.14. Suppose that $0 < \alpha < 2$. Then

$$\frac{\sin(\pi \mu (1 - \alpha))}{\sin(\pi \mu)} = -1,$$

(82)

if and only if

$$\mu = \frac{2n - 1}{\alpha},$$

(83)

or

$$\mu = \frac{2n}{2 - \alpha},$$

(84)

for some integer $n$. Likewise,

$$\frac{\sin(\pi \mu (1 - \alpha))}{\sin(\pi \mu)} = 1,$$

(85)

if and only if

$$\mu = \frac{2n}{\alpha},$$

(86)

or

$$\mu = \frac{2n - 1}{2 - \alpha},$$

(87)

for some integer $n$.

Theorem 3.15. Suppose that $0 < \alpha < 2$ and that $n$ and $m$ are positive integers. Then

$$\frac{2n}{2 - \alpha} = \frac{2m - 1}{\alpha},$$

(88)

if and only if

$$\alpha = \frac{4m - 2}{2n + 2m - 1}.$$
Proof. Suppose that

\[
\frac{2n}{2 - \alpha} = \frac{2m - 1}{\alpha}. \tag{90}
\]

Multiplying both sides of (90) by

\[
\frac{2 - \alpha}{2m - 1}, \tag{91}
\]

we have

\[
\frac{2n}{2m - 1} = \frac{2}{\alpha} - 1. \tag{92}
\]

Thus,

\[
\frac{2}{\alpha} = \frac{2n + 2m - 1}{2m - 1}, \tag{93}
\]

so

\[
\alpha = \frac{4m - 2}{2n + 2m - 1}. \tag{94}
\]

Corollary 3.16. Suppose that 0 < \(\alpha\) < 2 and that \(n\) and \(m\) are positive integers. Then

\[
\frac{2n}{\alpha} = \frac{2m - 1}{2 - \alpha}, \tag{95}
\]

if and only if

\[
\alpha = \frac{4n}{2n + 2m - 1}. \tag{96}
\]

Theorem 3.17. Suppose that 0 < \(\alpha\) < 2 and that \(n\) is a positive integer. Suppose further that

\[
\frac{2n}{2 - \alpha} = \frac{2m - 1}{\alpha}, \tag{97}
\]

for some positive integer \(m\). Then

\[
\frac{2n}{2 - \alpha} = n + m - \frac{1}{2}. \tag{98}
\]

Proof. By Theorem 3.15,

\[
\alpha = \frac{4m - 2}{2n + 2m - 1}. \tag{99}
\]

Therefore,

\[
\frac{2m - 1}{\alpha} = \frac{2n + 2m - 1}{2} = n + m - \frac{1}{2}. \tag{100}
\]

Corollary 3.18. Suppose that 0 < \(\alpha\) < 2 and that \(n\) is a positive integer. Suppose further that

\[
\frac{2n}{\alpha} = \frac{2m - 1}{2 - \alpha}. \tag{101}
\]
for some positive integer \(m\). Then

\[
\frac{2n}{\alpha} = n + m - \frac{1}{2}.
\] (102)

**Theorem 3.19.** Suppose that \(0 < \alpha < 2\). Suppose further that

\[
\mu = \frac{2n}{2 - \alpha},
\] (103)

where \(n\) is a positive integer. Then

\[
(1 - \alpha) \cos(\pi (1 - \alpha)\mu) - \cot(\pi \mu) \sin(\pi (1 - \alpha)\mu) = 0,
\] (104)

if and only if

\[
\frac{2n}{2 - \alpha} = \frac{2m - 1}{\alpha},
\] (105)

for some positive integer \(m\).

**Proof.** Suppose that

\[
\frac{2n}{2 - \alpha} = \frac{2m - 1}{\alpha},
\] (106)

for some positive integer \(m\). Then, by Theorem 3.17,

\[
\mu = n + m - \frac{1}{2},
\] (107)

so

\[
\cos(\pi \mu) = \cos(\pi (n + m - \frac{1}{2})) = 0.
\] (108)

Furthermore, by Theorem 3.15,

\[
\alpha = \frac{4m - 2}{2n + 2m - 1}.
\] (109)

Therefore,

\[
1 - \alpha = \frac{2n - 2m + 1}{2n + 2m - 1}.
\] (110)

Combining (110) and (107),

\[
(1 - \alpha)\mu = \frac{2n - 2m + 1}{2n + 2m - 1} \cdot \left( n + m - \frac{1}{2} \right) = n - m + \frac{1}{2},
\] (111)

so

\[
\cos(\pi (1 - \alpha)\mu) = \cos(\pi (n - m + \frac{1}{2})) = 0.
\] (112)

Thus, from (112) and (108), we have that

\[
(1 - \alpha) \cos(\pi (1 - \alpha)\mu) - \cot(\pi \mu) \sin(\pi (1 - \alpha)\mu) = 0.
\] (113)
For the converse, suppose now that
\[(1 - \alpha) \cos(\pi(1 - \alpha)\mu) - \cot(\pi\mu) \sin(\pi(1 - \alpha)\mu) = 0.\] (114)
By Theorem 3.14,
\[ \frac{\sin(\pi(1 - \alpha)\mu)}{\sin(\pi\mu)} = -1.\] (115)
Thus,
\[(1 - \alpha) \cos(\pi(1 - \alpha)\mu) + \cos(\pi\mu) = 0.\] (116)
Combining (116) and (103), we have that
\[(1 - \alpha) \cos(\pi(1 - \alpha)\frac{2n}{2-\alpha}) + \cos(\pi\frac{2n}{2-\alpha}) = 0.\] (117)
Since
\[\cos(\pi(1 - \alpha) \cdot \frac{2n}{2-\alpha}) = \cos(-\pi \frac{2n}{2-\alpha} + 2n\pi) = \cos(\pi \frac{2n}{2-\alpha}),\] (118)
it follows that
\[(1 - \alpha) \cos(\pi \frac{2n}{2-\alpha}) + \cos(\pi \frac{2n}{2-\alpha}) = 0,\] (119)
so
\[(2 - \alpha) \cdot \cos(\pi \frac{2n}{2-\alpha}) = 0.\] (120)
Thus,
\[\cos(\pi \frac{2n}{2-\alpha}) = 0.\] (121)
Clearly, (121) implies that
\[\frac{2n}{2-\alpha} = k + \frac{1}{2},\] (122)
for some integer \(k > n - 1\). Therefore
\[\frac{4n}{2-\alpha} = 2k + 1,\] (123)
and
\[2 - \alpha = \frac{4n}{2k + 1},\] (124)
so
\[\alpha = \frac{4k - 4n + 2}{2k + 1}.\] (125)
Letting \(m = k - n + 1\), we have
\[\alpha = \frac{4m - 2}{2n + 2m - 1},\] (126)
and \( m > 0 \), so, by Theorem 3.15,

\[
\frac{2n}{2 - \alpha} = \frac{2m - 1}{\alpha}. \quad \square
\]  

**Corollary 3.20.** Suppose that \( 0 < \alpha < 2 \). Suppose further that

\[
\mu = \frac{2n}{\alpha},
\]  

where \( n \) is a positive integer. Then

\[(1 - \alpha) \cos(\pi(1 - \alpha)\mu) - \cot(\pi\mu) \sin(\pi(1 - \alpha)\mu) = 0,\]  

if and only if

\[
\frac{2n}{\alpha} = \frac{2m - 1}{2 - \alpha},
\]  

for some positive integer \( m \).

**Theorem 3.21.** Suppose \( 0 < \alpha < 2 \) and that \( i \) and \( n \) are integers. If

\[
i = \frac{2n - 1}{\alpha},
\]  

then

\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi \nu)}{i - \frac{2n - 1}{\nu}} = -\pi \alpha.
\]  

Likewise, if

\[
i = \frac{2n}{2 - \alpha},
\]  

then

\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi \nu)}{i - \frac{2n}{2 - \nu}} = -\pi(2 - \alpha).
\]

**Proof.** Suppose that

\[
i = \frac{2n - 1}{\alpha}.
\]  

Then, by L'Hôpital's rule,

\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi \nu)}{i - \frac{2n - 1}{\nu}} = -\pi \cdot \lim_{\nu \to \alpha} \nu \cdot \frac{\sin(i\pi \nu - (2n - 1)\pi)}{i\pi \nu - (2n - 1)\pi} = -\pi \alpha.
\]  

Likewise, suppose that

\[
i = \frac{2n}{2 - \alpha}.
\]  

Then,

\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi \nu)}{i - \frac{2n}{2 - \nu}} = -\pi \cdot \lim_{\nu \to \alpha} (2 - \nu) \cdot \frac{\sin(i\pi (2 - \nu) - 2n\pi)}{i\pi (2 - \nu) - 2n\pi} = -\pi(2 - \alpha). \quad \square
\]
Theorem 3.22. Suppose $0 < \alpha < 2$ and that $i$ and $n$ are integers. If
\[
i = \frac{2n - 1}{2 - \alpha},
\]
then
\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2n - 1}{2 - \nu}} = \pi(2 - \alpha).
\]
Likewise, if
\[
i = \frac{2n}{\alpha},
\]
then
\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{i - \frac{2n}{2 - \nu}} = \pi\alpha.
\]

Theorem 3.23. Suppose that $0 < \alpha < 2$, and that $L$ and $M$ are arbitrary nonnegative integers. Suppose further that $p: (0, 2) \times \mathbb{Z} \to \mathbb{R}$ is defined by
\[
p(\nu, i) = \prod_{\ell=1}^{L} \left( i - \frac{2\ell - 1}{\nu} \right) \prod_{m=1}^{M} \left( i - \frac{2m}{2 - \nu} \right),
\]
for all $0 < \nu < 2$ and integers $i$. Then, for each integer $i$, the limit
\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)}
\]
extists and is finite.

Proof. Suppose that $i$ is an integer and that
\[
i = \frac{2\ell - 1}{\alpha},
\]
for some integer $1 \leq \ell \leq L$. By Theorem 3.17,
\[
\frac{2\ell - 1}{\alpha} \neq \frac{2m}{2 - \alpha},
\]
for all positive integers $m$, so, by Theorem 3.21,
\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)}
\]
extists and is finite.

Likewise, suppose that $i$ is an integer and that
\[
i = \frac{2m}{2 - \alpha},
\]
for some integer $1 \leq m \leq M$. By Theorem 3.17,
\[
\frac{2m}{2 - \alpha} \neq \frac{2\ell - 1}{\alpha},
\]

for all positive integers \(\ell\). Thus, by Theorem 3.21,
\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)}
\]

exists and is finite. \(\square\)

**Corollary 3.24.** Suppose that \(0 < \alpha < 2\), and that \(L\) and \(M\) are arbitrary nonnegative integers. Suppose further that \(p:\ (0,2) \times \mathbb{Z} \to \mathbb{R}\) is defined by
\[
p(\nu, i) = \prod_{\ell=1}^{L} \left(i - \frac{2\ell - 1}{2 - \nu}\right) \prod_{m=1}^{M} \left(i - \frac{2m}{\nu}\right),
\]

for all \(0 < \nu < 2\) and integers \(i\). Then, for each integer \(i\), the limit
\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi\nu)}{p(\nu, i)}
\]

exists and is finite.

**Theorem 3.25.** Suppose that \(x\) is a real number, and let \(\lfloor \cdot \rfloor\) and \(\lceil \cdot \rceil\) denote the floor and ceiling functions respectively. Then
\[
\lfloor x + \frac{1}{2} \rfloor \leq \lfloor x \rfloor.
\]

**Proof.** Let \(y = x - \lfloor x \rfloor\). Then \(0 \leq y < 1\), and
\[
\lfloor x + \frac{1}{2} \rfloor = \lfloor \lfloor x \rfloor + y + \frac{1}{2} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{1}{2} \rfloor.
\]

If \(0 \leq y < \frac{1}{2}\), then \(\lfloor y + \frac{1}{2} \rfloor = \lfloor y \rfloor\). On the other hand, if \(\frac{1}{2} \leq y < 1\), then \(\lfloor y + \frac{1}{2} \rfloor = \lfloor y \rfloor\). Thus,
\[
\lfloor y + \frac{1}{2} \rfloor \leq \lfloor y \rfloor.
\]

Combining (155) and (154),
\[
\lfloor x + \frac{1}{2} \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor = \lfloor x \rfloor. \quad \square
\]

4. Analytical apparatus

4.1. Integral identities

The following theorem is proved in [17], and is an identity involving the integral of a ratio of a polynomial and a non-integer power.

**Theorem 4.1.** Suppose that \(0 < \alpha < 2\), and that \(\text{Re} \mu > -1\) and \(\mu \neq 1, 2, 3, \ldots\). Then
\[
\int_{0}^{1} \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax \cos(\pi\alpha) + x^2} \, dx = \pi a^{\mu - 1} \frac{\sin(\mu\pi(1 - \alpha))}{\sin(\mu\pi)} + \sum_{k=0}^{\infty} \frac{\sin((k + 1)\pi\alpha)}{\mu - k - 1} a^k,
\]

for all \(0 < a < 1\).
The following theorem computes the integral in (157) when the integrand is multiplied by log(x). It is proved by differentiating (157) with respect to \( \mu \).

**Theorem 4.2.** Suppose that \( 0 < \alpha < 2 \), and that \( \Re(\mu) > -1 \) and \( \mu \neq 1, 2, 3, \ldots \). Then

\[
\int_0^1 \frac{x^\mu \log(x) \sin(\pi \alpha)}{a^2 - 2ax \cos(\pi \alpha) + x^2} \, dx = \pi a^{\mu-1} \log(a) \frac{\sin(\mu \pi (1-\alpha))}{\sin(\mu \pi)} \\
+ \frac{\pi^2 a^{\mu-1}}{\sin(\pi \mu)} \left( (1-\alpha) \cos(\pi (1-\alpha) \mu) - \cot(\pi \mu) \sin(\pi (1-\alpha) \mu) \right) \\
- \sum_{k=0}^{\infty} \frac{\sin((k+1)\pi \alpha)}{(\mu - k - 1)^2} a^k,
\]

(158)

for all \( 0 < a < 1 \).

### 4.2. The invertibility of a certain linear mapping

In this section, we show that a certain matrix is invertible. The principal result of this section is **Theorem 4.4**.

The following is a technical lemma involving the limits of a certain ratio of a sine function and a rational function.

**Lemma 4.3.** Suppose that \( N \) is a positive integer and that \( 0 < \alpha < 2 \). Letting \([\cdot]\) and \(\lceil \cdot \rceil\) denote the floor and ceiling functions respectively, suppose further that

\[
\mathcal{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor,
\]

(159)

and

\[
\mathcal{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil,
\]

(160)

and observe that \( \mathcal{L} + \mathcal{M} = N \). Suppose finally that \( p: (0,2) \times \mathbb{Z} \to \mathbb{R} \) is defined by

\[
p(\nu, i) = \prod_{j=1}^{\mathcal{L}} \left( i - \frac{2j-1}{\nu} \right) \prod_{k=1}^{\mathcal{M}} \left( i - \frac{2k}{2-\nu} \right),
\]

(161)

for all \( 0 < \nu < 2 \) and integers \( i \). Then, for each integer \( 1 \leq i \leq N \),

\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi \nu)}{p(\nu, i)}
\]

exists and is finite, and

\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi \nu)}{p(\nu, i)} \neq 0.
\]

(163)

**Proof.** Suppose that \( 1 \leq i \leq N \) is an integer. Then, by **Theorem 3.23**, the limit

\[
\lim_{\nu \to \alpha} \frac{\sin(i\pi \nu)}{p(\nu, i)}
\]

(164)
exists and is finite. Suppose further that $\alpha i$ is an integer. Then

$$\sin(i\pi \alpha) = 0,$$

(165)

and, furthermore,

$$\lim_{\nu \to \alpha} \frac{\sin(i\pi \nu)}{p(\nu, i)} \neq 0$$

(166)

if and only if either

$$i = \frac{2j - 1}{\alpha},$$

(167)

for some integer $j$, where

$$1 \leq j \leq \left\lfloor \frac{N\alpha}{2} \right\rfloor,$$

(168)

or

$$i = \frac{2k}{2 - \alpha},$$

(169)

for some integer $k$, where

$$1 \leq k \leq \left\lfloor \frac{(2 - \alpha)N}{2} \right\rfloor.$$

(170)

Suppose that $i\alpha$ is an odd integer. Then

$$i\alpha = 2j - 1,$$

(171)

for some integer $j$. Therefore,

$$i = \frac{2j - 1}{\alpha},$$

(172)

for some integer $j$. Observing that

$$j = \frac{\alpha i}{2} + \frac{1}{2},$$

(173)

and combining (173) with $1 \leq i \leq N$, we have

$$\frac{\alpha}{2} + \frac{1}{2} \leq j \leq \frac{N\alpha}{2} + \frac{1}{2},$$

(174)

so

$$1 \leq j \leq \left\lfloor \frac{N\alpha}{2} + \frac{1}{2} \right\rfloor.$$

(175)

Thus, by Theorem 3.25,

$$1 \leq j \leq \left\lfloor \frac{N\alpha}{2} \right\rfloor.$$  

(176)

Suppose now that $i\alpha$ is an even integer. Then

$$i(2 - \alpha) = 2k,$$

(177)
for some integer $k$. Therefore,
\[ i = \frac{2k}{2 - \alpha}, \tag{178} \]
for some integer $k$. Observing that
\[ k = \frac{(2 - \alpha)i}{2}, \tag{179} \]
and combining (179) with $1 \leq i \leq N$, we have
\[ \frac{2 - \alpha}{2} \leq k \leq \frac{(2 - \alpha)N}{2}. \tag{180} \]
Thus,
\[ 1 \leq k \leq \left\lfloor \frac{(2 - \alpha)N}{2} \right\rfloor. \tag{181} \]

The following theorem is the principal result of this section.

**Theorem 4.4.** Suppose that $0 < \alpha < 2$. Suppose further that $N$ is a positive integer, and, letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that
\[ L = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{182} \]
and
\[ M = \left\lfloor \frac{(2 - \alpha)N}{2} \right\rfloor, \tag{183} \]
and observe that $L + M = N$. Suppose finally that $A$ is the $N \times N$ matrix defined by the formula
\[ A_{i,j} = \lim_{\nu \to \alpha} \frac{\sin(i \pi \nu)}{i - \frac{2j - 1}{\nu}}, \tag{184} \]
\[ A_{i,L+k} = \begin{cases} 
\lim_{\nu \to \alpha} \frac{\sin(i \pi \nu)}{i - \frac{2k}{2 - \alpha}} & \text{if } \frac{2k}{2 - \alpha} \neq \frac{2\ell - 1}{\alpha} \text{ for all integers } 1 \leq \ell \leq L, \\
\frac{\sin(i \pi \alpha)}{(i - \frac{2k}{2 - \alpha})^2} & \text{if } \frac{2k}{2 - \alpha} = \frac{2\ell - 1}{\alpha} \text{ for some integer } 1 \leq \ell \leq L, 
\end{cases} \tag{185} \]
for all integers $1 \leq i \leq N$, $1 \leq j \leq L$ and $1 \leq k \leq M$ (Theorems 3.21 and 3.17 show that matrix $A$ is well-defined). Then matrix $A$ is invertible.

**Proof.** Suppose that $p: (0, 2) \times \mathbb{Z} \to \mathbb{R}$ is defined by
\[ p(\nu, i) = \prod_{j=1}^{T} \left( i - \frac{2j - 1}{\nu} \right) \prod_{k=1}^{M} \left( i - \frac{2k}{2 - \nu} \right), \tag{186} \]
for all $0 < \nu < 2$ and integers $i$. Suppose further that $D$ is the diagonal $N \times N$ matrix defined by
\[ D_{i,j} = \begin{cases} 
\lim_{\nu \to \alpha} \frac{\sin(i \pi \nu)}{p(\nu, i)} & \text{if } i = j, \\
0 & \text{if } i \neq j, 
\end{cases} \tag{187} \]
for all integers \(1 \leq i, j \leq N\). By Lemma 4.3, the matrix \(D\) is well-defined and invertible. Therefore,

\[
(D^{-1} A)_{i,j} = \lim_{\nu \to \alpha} \frac{p(\nu, i)}{1 - \frac{2j - 1}{\nu}},
\]

(188)

\[
(D^{-1} A)_{i,\ell+k} = \begin{cases} 
\lim_{\nu \to \alpha} \frac{p(\nu, i)}{1 - \frac{2k}{\nu}} & \text{if } \frac{2k}{2-\alpha} \neq \frac{2\ell-1}{\alpha} \text{ for all integers } 1 \leq \ell \leq \ell,
(p(\alpha, i)) \frac{(i - \frac{2k}{2-\alpha})^2}{(i - \frac{2k}{2-\alpha})^2} & \text{if } \frac{2k}{2-\alpha} = \frac{2\ell-1}{\alpha} \text{ for some integer } 1 \leq \ell \leq \ell,
\end{cases}
\]

(189)

for all integers \(1 \leq i \leq N, 1 \leq j \leq \ell\) and \(1 \leq k \leq M\). By Theorem 3.13, the matrix \(D^{-1} A\) is invertible and, since \(D\) is invertible, so is the matrix \(A\). \(\Box\)

5. Analysis of the integral equation: the Neumann case

Suppose that the curve \(\gamma: [-1, 1] \to \mathbb{R}^2\) is a wedge defined by (58) with interior angle \(\pi \alpha\), where \(0 < \alpha < 2\) (see Fig. 4), and let \(\nu(t)\) denote the inward-facing unit normal to the curve \(\gamma\) at the point \(\gamma(t)\). Let \(g\) be a function in \(L^2([-1, 1])\), and suppose that \(\rho \in L^2([-1, 1])\) solves the equation

\[
-\pi \rho(s) + \int_{-1}^{1} \psi^1_{\gamma(s),\nu(s)}(\gamma(t)) \rho(t) \, dt = g(s),
\]

(190)

for all \(s \in [-1, 1]\), where \(\psi^1_{\gamma(s),\nu(s)}\) is defined by (43).

In this section, we analyze this boundary integral equation, which is well-posed even though the curve \(\gamma\) is open (see Observation 3.2). We investigate the behavior of (190) for functions \(\rho \in L^2([-1, 1])\) of the forms

\[
\rho(t) = |t|^\mu - 1,
\]

(191)

\[
\rho(t) = |t|^\mu \log(|t|),
\]

(192)

\[
\rho(t) = \text{sgn}(t)|t|^\mu - 1,
\]

(193)

\[
\rho(t) = \text{sgn}(t)|t|^\mu \log(|t|),
\]

(194)

where \(\mu > \frac{1}{2}\) is a real number. If \(\rho\) has the forms (191), (192), (193), (194), then for most values of \(\mu\) the resulting \(g\) is singular. In Section 5.1, we observe that for certain \(\mu\), the function \(g\) is smooth. In Section 5.2, we fix \(g\) and view (190) as an integral equation in \(\rho\). We then observe that if \(g\) is smooth, then the solution \(\rho\) is representable by a series of functions of the forms (191), (192), (193), (194).

5.1. The singularities in the solution of equation (190)

In this section we observe that for certain functions \(\rho\), the function \(g\) is representable by convergent Taylor series on the intervals \([-1, 0]\) and \([0, 1]\).

5.1.1. The even case

Suppose that \(\rho \in L^2([-1, 1])\) is an even function, and suppose that \(g \in L^2([-1, 1])\) is defined by (190). By Theorem 3.10, \(g\) is also even and

\[
g(s) = -\pi \rho(s) - \int_{0}^{1} \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt,
\]

(195)

for all \(0 < s \leq 1\).
Suppose that $\rho(t) = t^{\mu-1}$ for all $0 \leq t \leq 1$. The following theorem shows that for certain values of $\mu$, the function $g$ in (195) is representable by a convergent Taylor series on the interval $[0,1]$. This theorem is proved in [17]; a proof is provided here for completeness.

**Theorem 5.1.** Suppose that $0 < \alpha < 2$ is a real number and $n$ is a positive integer. Then

\[
\pi s^{\frac{2n-1}{\alpha}} - 1 + \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{\frac{2n-1}{\alpha}} dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \sin(m\pi \nu) \right) s^{m-1},
\]

(196)

\[
\pi s^{\frac{2n}{\alpha}} - 1 + \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{\frac{2n}{\alpha}} dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \sin(m\pi \nu) \right) s^{m-1},
\]

(197)

for all $0 < s \leq 1$.

**Proof.** Substituting

\[
\mu = \frac{2n - 1}{\alpha}
\]

(198)

into identity (157) and applying Theorem 3.14, we have

\[
\pi s^{\frac{2n-1}{\alpha}} - 1 + \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{\frac{2n-1}{\alpha}} dt = -\sum_{m=1}^{\infty} \sin(m\pi \alpha) s^{m-1},
\]

(199)

for all $0 < \alpha < 2$ such that

\[
\frac{2n - 1}{\alpha} \neq 1, 2, 3, \ldots
\]

(200)

Viewing both sides of (199) as functions of $\alpha$, we observe that Theorem 3.21 extends identity (199) to all $0 < \alpha < 2$. The proof of identity (197) is essentially identical. \(\square\)

Suppose now that $\rho(t) = t^{\mu-1} \log(t)$ for all $0 \leq t \leq 1$. The following theorem shows that, for certain values of $\mu$, the function $g$ in (195) is representable by a convergent Taylor series on the interval $[0,1]$.

**Theorem 5.2.** Suppose that $0 < \alpha < 2$ is a real number and $n$ is a positive integer. Suppose further that

\[
\frac{2n}{2 - \alpha} = \frac{2k - 1}{\alpha},
\]

(201)

for some positive integer $k$. Then

\[
\pi s^{\frac{2n}{2 - \alpha}} \log(s) + \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{\frac{2n}{2 - \alpha}} \log(t) dt = -\sum_{m=1}^{\infty} \frac{\sin(m\pi \alpha)}{(m - \frac{2n}{2 - \alpha})^2} s^{m-1},
\]

(202)

for all $0 < s \leq 1$.

**Proof.** Identity (202) follows from combining

\[
\mu = \frac{2n}{2 - \alpha}
\]

(203)
and (201) with (158) and Theorems 3.14 and 3.19. By Theorem 3.17 and (201),

\[ \frac{2n}{2-\alpha} = n + k - \frac{1}{2}, \]  

(204)

so the right hand side of (202) is well-defined. □

5.1.2. The odd case

Suppose that \( \rho \in L^2([-1, 1]) \) is an odd function, and suppose that \( g \in L^2([-1, 1]) \) is defined by (190). By Theorem 3.10, \( g \) is also odd and

\[ g(s) = -\pi \rho(s) + \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt, \]  

(205)

for all \( 0 < s \leq 1 \).

Suppose that \( \rho(t) = t^{\mu-1} \) for all \( 0 \leq t \leq 1 \). The following theorem shows that for certain values of \( \mu \), the function \( g \) in (205) is representable by a convergent Taylor series on the interval \([0, 1]\).

**Theorem 5.3.** Suppose that \( 0 < \alpha < 2 \) is a real number and \( n \) is a positive integer. Then

\[ -\pi s^{\frac{2n}{2-\alpha} - 1} + \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{\frac{2n}{2-\alpha} - 1} \, dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(m \pi \nu)}{m - \frac{2n-1}{2-\alpha}} \right) s^{m-1}, \]  

(206)

\[ -\pi s^{\frac{2n}{2-\alpha} - 1} + \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{\frac{2n}{2-\alpha} - 1} \, dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(m \pi \nu)}{m - \frac{2n}{2-\alpha}} \right) s^{m-1}, \]  

(207)

for all \( 0 < s \leq 1 \).

Suppose now that \( \rho(t) = t^{\mu-1} \log(t) \) for all \( 0 \leq t \leq 1 \). The following theorem shows that, for certain values of \( \mu \), the function \( g \) in (205) is representable by a convergent Taylor series on the interval \([0, 1]\).

**Theorem 5.4.** Suppose that \( 0 < \alpha < 2 \) is a real number and \( n \) is a positive integer. Suppose further that

\[ \frac{2n}{\alpha} = \frac{2k-1}{2-\alpha}, \]  

(208)

for some positive integer \( k \). Then

\[ -\pi s^{\frac{2n}{\alpha} - 1} \log(s) + \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{\frac{2n}{\alpha} - 1} \log(t) \, dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(m \pi \alpha)}{m - \frac{2n}{\alpha}} \right) s^{m-1}, \]  

(209)

for all \( 0 < s \leq 1 \).

5.2. Series representation of the solution of equation (190)

Suppose that \( g \) is representable by convergent Taylor series on the intervals \([-1, 0]\) and \([0, 1]\). Suppose further that \( \rho \) solves equation (190). In this section we observe that \( \rho \) is representable by a linear combination of certain non-integer powers and non-integer powers multiplied by logarithms, on the intervals \([-1, 0]\) and \([0, 1]\).
5.2.1. The even case

Suppose that \( g \in L^2([-1, 1]) \) is an even function, and suppose that \( \rho \in L^2([-1, 1]) \) satisfies equation (190). By Theorem 3.10, \( \rho \) is also even and

\[
-\pi \rho(s) - \int_0^1 \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt = g(s),
\]

for all \( 0 < s \leq 1 \), where \( 0 < \alpha < 2 \).

Suppose that \( N \) is a positive integer. Letting \( [\cdot] \) and \( \lceil \cdot \rceil \) denote the floor and ceiling functions respectively, suppose that

\[
L = \left\lfloor \frac{\alpha N}{2} \right\rfloor,
\]

and

\[
M = \left\lceil \frac{(2 - \alpha)N}{2} \right\rceil,
\]

and observe that \( L + M = N \). The following theorem shows that if \( g \) is representable by a convergent Taylor series on \( [0, 1] \), then there exist real numbers \( b_1, b_2, \ldots, b_N \) such that the function

\[
\rho(t) = \sum_{i=1}^{T} b_i t^{\frac{2i-1}{\alpha} - 1} + \sum_{i=0}^{M} b_{L+i} t^{\frac{2i}{\alpha} - 1 - 1} + \sum_{i=0}^{M} b_{L+i} t^{\frac{2i}{\alpha} - 1 - 1} \log(t),
\]

where \( 0 < t \leq 1 \) and

\[
S = \left\{ i \in \mathbb{Z}, 1 \leq i \leq M : \frac{2i}{2 - \alpha} = \frac{2j - 1}{\alpha} \text{ for some integer } 1 \leq j \leq L \right\},
\]

solves equation (210) to within an error \( O(t^N) \).

**Theorem 5.5.** Suppose that \( 0 < \alpha < 2 \) and that \( N \) is a positive integer. Letting \( [\cdot] \) and \( \lceil \cdot \rceil \) denote the floor and ceiling functions respectively, suppose that

\[
L = \left\lfloor \frac{\alpha N}{2} \right\rfloor,
\]

and

\[
M = \left\lceil \frac{(2 - \alpha)N}{2} \right\rceil,
\]

and observe that \( L + M = N \). Suppose that \( g : [0, 1] \to \mathbb{R} \) is defined by the formula

\[
g(t) = \sum_{i=0}^{\infty} c_{i+1} t^i,
\]

for all \( 0 \leq t \leq 1 \), where \( c_1, c_2, \ldots \) are real numbers. Then, for any \( 0 < \alpha < 2 \), there exist unique real numbers \( b_1, b_2, \ldots, b_N \) such that if

\[
\rho(t) = \sum_{i=1}^{T} b_i t^{\frac{2i-1}{\alpha} - 1} + \sum_{i=0}^{M} b_{L+i} t^{\frac{2i}{\alpha} - 1 - 1} + \sum_{i=0}^{M} b_{L+i} t^{\frac{2i}{\alpha} - 1 - 1} \log(t),
\]

for all \( 0 \leq t \leq 1 \).
where $0 < t \leq 1$ and
\[
S = \left\{ i \in \mathbb{Z}, 1 \leq i \leq M : \frac{2i}{2 - \alpha} = \frac{2j - 1}{\alpha} \text{ for some integer } 1 \leq j \leq T \right\}, \quad (219)
\]
then
\[
- \pi \rho(s) - \int_{0}^{\frac{1}{2}} \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt = g(s) + \sum_{i=N}^{\infty} d_i s^i, \quad (220)
\]
for all $0 < s \leq 1$, for some real numbers $d_N, d_{N+1}, \ldots$. 

**Proof.** By Theorem 4.4, the $N \times N$ matrix $A$, defined by (184) and (185), is invertible for any $0 < \alpha < 2$. Therefore, there exist unique real numbers $b_1, b_2, \ldots, b_N$ such that
\[
\sum_{j=1}^{n} A_{i,j} b_j = c_i, \quad (221)
\]
for each $i = 1, 2, \ldots, N$. Suppose that $\rho : [0, 1] \to \mathbb{R}$ is defined by (218). By Theorems 5.1 and 5.2,
\[
- \pi \rho(s) - \int_{0}^{\frac{1}{2}} \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt
\]
\[
= \sum_{j=1}^{T} b_j \sum_{i=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(i \pi \nu)}{i - \frac{2j - 1}{2 - \alpha}} \right) s^{i-1} + \sum_{j=1}^{M} b_{T+j} \sum_{i=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(i \pi \nu)}{i - \frac{2j - 1}{2 - \alpha}} \right) s^{i-1}
\]
\[
+ \sum_{j=1}^{M} b_{T+j} \sum_{i=1}^{N} \frac{\sin(i \pi \nu)}{i - \frac{2j - 1}{2 - \alpha}} s^{i-1} + \sum_{i=N}^{\infty} d_i s^i, \quad (222)
\]
where $0 < s \leq 1$ and $S$ is defined by (219), for some real numbers $d_N, d_{N+1}, \ldots$. Therefore,
\[
- \pi \rho(s) - \int_{0}^{\frac{1}{2}} \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt
\]
\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} A_{i,j} b_j s^{i-1} + \sum_{i=N}^{\infty} d_i s^i
\]
\[
= \sum_{i=1}^{N} c_i s^{i-1} + \sum_{i=N}^{\infty} d_i s^i = g(s) + \sum_{i=N}^{\infty} (d_i - c_{i+1}) s^i, \quad (223)
\]
for all $0 < s \leq 1$. □
5.2.2. The odd case

Suppose that $g \in L^2([-1, 1])$ is an odd function, and suppose that $\rho \in L^2([-1, 1])$ satisfies equation (190). By Theorem 3.10, $\rho$ is also odd and

$$- \pi \rho(s) + \int_{0}^{1} \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt = g(s),$$

(224)

for all $0 < s \leq 1$, where $0 < \alpha < 2$.

Suppose that $N$ is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that

$$\overline{M} = \left\lfloor \frac{(2 - \alpha)N}{2} \right\rfloor,$$

(225)

and

$$L = \left\lceil \frac{\alpha N}{2} \right\rceil,$$

(226)

and observe that $\overline{M} + L = N$. The following theorem shows that if $g$ is representable by a convergent Taylor series on $[0, 1]$, then there exist real numbers $b_1, b_2, \ldots, b_N$ such that the function

$$\rho(t) = \sum_{i=1}^{\overline{M}} b_i t^{2i-1} + \sum_{i \in P}^{L} b_{2i+1} t^{2i-1} + \sum_{i \in \overline{P}}^{L} b_{2i+1} t^{2i-1} \log(t),$$

(227)

where $0 < t \leq 1$ and

$$T = \left\{ i \in \mathbb{Z}, 1 \leq i \leq L : \frac{2i - 1}{\alpha} = \frac{2j - 1}{2 - \alpha} \quad \text{for some integer } 1 \leq j \leq \overline{M} \right\},$$

(228)

solves equation (224) to within an error $O(t^N)$.

**Theorem 5.6.** Suppose that $0 < \alpha < 2$ and that $N$ is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that

$$\overline{M} = \left\lfloor \frac{(2 - \alpha)N}{2} \right\rfloor,$$

(229)

and

$$L = \left\lceil \frac{\alpha N}{2} \right\rceil,$$

(230)

and observe that $\overline{M} + L = N$. Suppose that $g : [0, 1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \sum_{i=0}^{\infty} c_{i+1} t^i,$$

(231)

for all $0 \leq t \leq 1$, where $c_1, c_2, \ldots$ are real numbers. Then, for any $0 < \alpha < 2$, there exist unique real numbers $b_1, b_2, \ldots, b_N$ such that if

$$\rho(t) = \sum_{i=1}^{\overline{M}} b_i t^{2i-1} + \sum_{i \in P}^{L} b_{2i+1} t^{2i-1} + \sum_{i \in \overline{P}}^{L} b_{2i+1} t^{2i-1} \log(t),$$

(232)
where }0 < t \leq 1\text{ and }
\begin{align}
T = \left\{ i \in \mathbb{Z}, 1 \leq i \leq L : \frac{2i}{\alpha} = \frac{2j - 1}{2 - \alpha} \text{ for some integer } 1 \leq j \leq M \right\}, \tag{233}
\end{align}
then
\begin{align}
- \pi \rho(s) + \int_{0}^{1} \frac{t \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt = g(s) + \sum_{i=N}^{\infty} d_i s^i, \tag{234}
\end{align}
for all }0 < s \leq 1,\text{ for some real numbers }d_N, d_{N+1}, \ldots.

6. Analysis of the integral equation: the Dirichlet case

Suppose that the curve }\gamma: [-1, 1] \to \mathbb{R}^2\text{ is a wedge defined by (58) with interior angle }\pi \alpha,\text{ where }0 < \alpha < 2\text{ (see Fig. 4), and let }\nu(t)\text{ denote the inward-facing unit normal to the curve }\gamma\text{ at the point }\gamma(t)\text{. Let }g\text{ be a function in }L^2([-1, 1])\text{, and suppose that }\rho \in L^2([-1, 1])\text{ solves the equation}
\begin{align}
- \pi \rho(s) + \int_{-1}^{1} \psi_{\gamma(t), \nu(t)}^{1}(\gamma(s)) \rho(t) \, dt = g(s), \tag{235}
\end{align}
for all }s \in [-1, 1],\text{ where }\psi_{\gamma(t), \nu(t)}^{1}\text{ is defined by (43).}
In this section, we analyze this boundary integral equation, which is well-posed even though the curve }\gamma\text{ is open (see Observation 3.2). We investigate the behavior of (235) for functions }\rho \in L^2([-1, 1])\text{ of the forms}
\begin{align}
\rho(t) &= |t|^\mu, \tag{236}
\rho(t) &= |t|^\mu \log(|t|), \tag{237}
\rho(t) &= \text{sgn}(t)|t|^\mu, \tag{238}
\rho(t) &= \text{sgn}(t)|t|^\mu \log(|t|), \tag{239}
\end{align}
where }\mu > \frac{1}{2}\text{ is a real number. If }\rho\text{ has the forms (236), (237), (238), (239), then for most values of }\mu\text{ the resulting }g\text{ is singular. In Section 6.1, we observe that for certain }\mu\text{, the function }g\text{ is smooth. In Section 6.2, we fix }g\text{ and view (235) as an integral equation in }\rho\text{. We then observe that if }g\text{ is smooth, then the solution }\rho\text{ is representable by a series of functions of the forms (236), (237), (238), (239).}

6.1. The singularities in the solution of equation (235)

In this section we observe that for certain functions }\rho,\text{ the function }g\text{ is representable by convergent Taylor series on the intervals }[-1, 0]\text{ and }[0, 1].

6.1.1. The even case

Suppose that }\rho \in L^2([-1, 1])\text{ is an even function, and suppose that }g \in L^2([-1, 1])\text{ is defined by (235). By Theorem 3.12, }g\text{ is also even and}
\begin{align}
g(s) = -\pi \rho(s) - \int_{0}^{1} \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt, \tag{240}
\end{align}
for all }0 < s \leq 1.
Suppose that \( \rho(t) = t^\mu \) for all \( 0 \leq t \leq 1 \). The following theorem shows that for certain values of \( \mu \), the function \( g \) in (240) is representable by a convergent Taylor series on the interval \([0, 1]\).

**Theorem 6.1.** Suppose that \( 0 < \alpha < 2 \) is a real number and \( n \) is a positive integer. Then

\[
\pi + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \, dt = (2 - \alpha)\pi - \sum_{m=1}^{\infty} \frac{\sin(m\pi \alpha)}{m} s^m, \tag{241}
\]

\[
\pi s^{2n-1} + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{2n-1} \, dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(m\pi \nu)}{m - \frac{2n}{2-\nu}} \right) s^m, \tag{242}
\]

\[
\pi s^{2n} + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{2n} \, dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(m\pi \nu)}{m - \frac{2n}{2-\nu}} \right) s^m, \tag{243}
\]

for all \( 0 < s \leq 1 \).

Suppose now that \( \rho(t) = t^\mu \log(t) \) for all \( 0 \leq t \leq 1 \). The following theorem shows that, for certain values of \( \mu \), the function \( g \) in (240) is representable by a convergent Taylor series on the interval \([0, 1]\).

**Theorem 6.2.** Suppose that \( 0 < \alpha < 2 \) is a real number and \( n \) is a positive integer. Suppose further that

\[
\frac{2n}{2-\alpha} = \frac{2k-1}{\alpha}, \tag{244}
\]

for some positive integer \( k \). Then

\[
\pi s^{2n} \log(s) + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{2n} \log(t) \, dt = -\sum_{m=1}^{\infty} \frac{\sin(m\pi \alpha)}{m - \frac{2n}{2-\alpha}} s^m, \tag{245}
\]

for all \( 0 < s \leq 1 \).

6.1.2. The odd case

Suppose that \( \rho \in L^2([-1, 1]) \) is an odd function, and suppose that \( g \in L^2([-1, 1]) \) is defined by (235). By Theorem 3.12, \( g \) is also odd and

\[
g(s) = -\pi \rho(s) + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt, \tag{246}
\]

for all \( 0 < s \leq 1 \).

Suppose that \( \rho(t) = t^\mu \) for all \( 0 \leq t \leq 1 \). The following theorem shows that for certain values of \( \mu \), the function \( g \) in (246) is representable by a convergent Taylor series on the interval \([0, 1]\).

**Theorem 6.3.** Suppose that \( 0 < \alpha < 2 \) is a real number and \( n \) is a positive integer. Then

\[
-\pi + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \, dt = -\alpha \pi - \sum_{m=1}^{\infty} \frac{\sin(m\pi \alpha)}{m} s^m, \tag{247}
\]
\[-\pi s^{2n-1} + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{2n-1} \, dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(m \pi \nu)}{m - \frac{2n-1}{2}} \right) s^m, \quad (248)\]

\[-\pi s^{2n} + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{2n} \, dt = -\sum_{m=1}^{\infty} \left( \lim_{\nu \to \alpha} \frac{\sin(m \pi \nu)}{m - \frac{2n}{2}} \right) s^m, \quad (249)\]

for all \(0 < s \leq 1\).

Suppose now that \(\rho(t) = t^\mu \log(t)\) for all \(0 \leq t \leq 1\). The following theorem shows that, for certain values of \(\mu\), the function \(g\) in (246) is representable by a convergent Taylor series on the interval \([0, 1]\).

**Theorem 6.4.** Suppose that \(0 < \alpha < 2\) is a real number and \(n\) is a positive integer. Suppose further that

\[\frac{2n}{\alpha} = \frac{2k - 1}{2 - \alpha}, \quad (250)\]

for some positive integer \(k\). Then

\[-\pi s^{2n} \log(s) + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} t^{2n} \log(t) \, dt = -\sum_{m=1}^{\infty} \frac{\sin(m \pi \alpha)}{m - \frac{2n}{\alpha}} s^m, \quad (251)\]

for all \(0 < s \leq 1\).

### 6.2. Series representation of the solution of Equation (235)

Suppose that \(g\) is representable by convergent Taylor series on the intervals \([-1, 0]\) and \([0, 1]\). Suppose further that \(\rho\) solves equation (235). In this section we observe that \(\rho\) is representable by a linear combination of certain non-integer powers and non-integer powers multiplied by logarithms, on the intervals \([-1, 0]\) and \([0, 1]\).

**6.2.1. The even case**

Suppose that \(g \in L^2([-1, 1])\) is an even function, and suppose that \(\rho \in L^2([-1, 1])\) satisfies equation (235). By **Theorem 3.12**, \(\rho\) is also even and

\[-\pi \rho(s) - \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt = g(s), \quad (252)\]

for all \(0 < s \leq 1\), where \(0 < \alpha < 2\).

Suppose that \(N\) is a positive integer. Letting \([\cdot]\) and \(\lceil \cdot \rceil\) denote the floor and ceiling functions respectively, suppose that

\[\ell = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \quad (253)\]

and

\[M = \left\lceil \frac{(2 - \alpha)N}{2} \right\rceil, \quad (254)\]

and observe that \(\ell + M = N\). The following theorem shows that if \(g\) is representable by a convergent Taylor series on \([0, 1]\), then there exist real numbers \(b_0, b_1, \ldots, b_N\) such that the function
\[
\rho(t) = b_0 + \sum_{i=1}^{T} b_i t^{\frac{2i-1}{\alpha}} + \sum_{i\in S}^{M} b_{L+i} t^{\frac{2i}{\alpha}} + \sum_{i\in \mathbb{Z}}^{M} b_{L+i} t^{\frac{2i}{\alpha}} \log(t),
\]
(255)

where \(0 < t \leq 1\) and

\[
S = \left\{ i \in \mathbb{Z}, 1 \leq i \leq M : \frac{2i}{2 - \alpha} = \frac{2j - 1}{\alpha} \text{ for some integer } 1 \leq j \leq L \right\},
\]
(256)
solves equation (252) to within an error \(O(t^{N+1})\).

**Theorem 6.5.** Suppose that \(0 < \alpha < 2\) and that \(N\) is a positive integer. Letting \([\cdot]\) and \(\lceil \cdot \rceil\) denote the floor and ceiling functions respectively, suppose that

\[
L = \left\lfloor \frac{\alpha N}{2} \right\rfloor,
\]
(257)

and

\[
M = \left\lceil \frac{(2 - \alpha)N}{2} \right\rceil,
\]
(258)

and observe that \(L + M = N\). Suppose that \(g : [0, 1] \to \mathbb{R}\) is defined by the formula

\[
g(t) = \sum_{i=0}^{\infty} c_i t^i,
\]
(259)

for all \(0 \leq t \leq 1\), where \(c_0, c_1, \ldots\) are real numbers. Then, for any \(0 < \alpha < 2\), there exist unique real numbers \(b_0, b_1, \ldots, b_N\) such that if

\[
\rho(t) = b_0 + \sum_{i=1}^{T} b_i t^{\frac{2i-1}{\alpha}} + \sum_{i\in S}^{M} b_{L+i} t^{\frac{2i}{\alpha}} + \sum_{i\in \mathbb{Z}}^{M} b_{L+i} t^{\frac{2i}{\alpha}} \log(t),
\]
(260)

where \(0 < t \leq 1\) and

\[
S = \left\{ i \in \mathbb{Z}, 1 \leq i \leq M : \frac{2i}{2 - \alpha} = \frac{2j - 1}{\alpha} \text{ for some integer } 1 \leq j \leq L \right\},
\]
(261)

then

\[
-\pi \rho(s) - \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) dt = g(s) + \sum_{i=N+1}^{\infty} d_i s^i,
\]
(262)

for all \(0 < s \leq 1\), for some real numbers \(d_{N+1}, d_{N+2}, \ldots\).

**6.2.2. The odd case**

Suppose that \(g \in L^2([-1, 1])\) is an odd function, and suppose that \(\rho \in L^2([-1, 1])\) satisfies equation (235). By **Theorem 3.12**, \(\rho\) is also odd and

\[
-\pi \rho(s) + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) dt = g(s),
\]
(263)

for all \(0 < s \leq 1\), where \(0 < \alpha < 2\).
Suppose that $N$ is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that
\[
M = \left\lfloor \frac{(2 - \alpha)N}{2} \right\rfloor,
\] (264)
and
\[
L = \left\lceil \frac{\alpha N}{2} \right\rceil,
\] (265)
and observe that $M + L = N$. The following theorem shows that if $g$ is representable by a convergent Taylor series on $[0, 1]$, then there exist real numbers $b_0, b_1, \ldots, b_N$ such that the function
\[
\rho(t) = b_0 + \sum_{i=1}^{M} b_i t^{\frac{2i-1}{2} - \alpha} + \sum_{i \in T} b_{M+i} t^{\frac{2i}{2} - \alpha} + \sum_{i \in T} b_{M+i} t^{\frac{2i}{2} - \alpha} \log(t),
\] (266)
where $0 < t < 1$ and
\[
T = \left\{ i \in \mathbb{Z}, 1 \leq i \leq L : \frac{2i - 1}{\alpha} = \frac{2j - 1}{2 - \alpha} \text{ for some integer } 1 \leq j \leq M \right\},
\] (267)
solves equation (263) to within an error $O(t^{N+1})$.

**Theorem 6.6.** Suppose that $0 < \alpha < 2$ and that $N$ is a positive integer. Letting $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, suppose that
\[
M = \left\lfloor \frac{(2 - \alpha)N}{2} \right\rfloor,
\] (268)
and
\[
L = \left\lceil \frac{\alpha N}{2} \right\rceil,
\] (269)
and observe that $M + L = N$. Suppose that $g: [0, 1] \rightarrow \mathbb{R}$ is defined by the formula
\[
g(t) = \sum_{i=0}^{\infty} c_i t^i,
\] (270)
for all $0 \leq t \leq 1$, where $c_0, c_1, \ldots$ are real numbers. Then, for any $0 < \alpha < 2$, there exist unique real numbers $b_0, b_1, \ldots, b_N$ such that if
\[
\rho(t) = b_0 + \sum_{i=1}^{M} b_i t^{\frac{2i-1}{2} - \alpha} + \sum_{i \in T} b_{M+i} t^{\frac{2i}{2} - \alpha} + \sum_{i \in T} b_{M+i} t^{\frac{2i}{2} - \alpha} \log(t),
\] (271)
where $0 < t \leq 1$ and
\[
T = \left\{ i \in \mathbb{Z}, 1 \leq i \leq L : \frac{2i - 1}{\alpha} = \frac{2j - 1}{2 - \alpha} \text{ for some integer } 1 \leq j \leq M \right\},
\] (272)
then

\[-\pi p(s) + \int_0^1 \frac{s \sin(\pi \alpha)}{s^2 + t^2 - 2st \cos(\pi \alpha)} \rho(t) \, dt = g(s) + \sum_{i=N+1}^{\infty} d_i s^i, \quad (273)\]

for all \(0 < s \leq 1\), for some real numbers \(d_{N+1}, d_{N+2}, \ldots\).

7. Extensions and generalizations

The apparatus of this paper extends to the Helmholtz equation on polygonal domains. It turns out that the solution to the Helmholtz equation with wavenumber \(k\) is representable by a formula essentially identical to, for example, (29), except with terms of the form \(|t|^\mu\) replaced by \(J_\mu(k|t|)\) and terms of the form \(|t|^\mu \log(|t|)\) replaced by \(\frac{\partial}{\partial \mu} J_\mu(k|t|)\). The proof follows the same lines as the proof presented in this paper: the matrix mapping the coefficients of the singular basis functions to the coefficients of the smooth Taylor series representing the boundary data is shown to be invertible for all angles \(\pi \alpha\). However, instead of using the properties of Lagrange interpolating polynomials, the proof uses properties of Bessel functions and their roots.

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Appendix A

In this section we provide a proof of Theorem 3.13, which is restated here as Theorem A.1.

The following theorem states that the Lagrange interpolating polynomials form a Chebyshev system (see Definition 3.1).

**Theorem A.1.** Suppose that \(K\) and \(L\) are nonnegative integers such that \(K \leq L\). Suppose further that \(\mu_1, \mu_2, \ldots, \mu_L\) are distinct real numbers. Suppose finally that \(p: \mathbb{R} \to \mathbb{R}\) is defined by

\[p(x) = \prod_{\ell=1}^{L} (x - \mu_\ell) \prod_{k=1}^{K} (x - \mu_k), \quad (274)\]

for all \(x \in \mathbb{R}\). Then the set of \(L + K\) functions

\[\left\{ \frac{p(x)}{x - \mu_\ell} \right\}_{\ell=1}^{L} \cup \left\{ \frac{p(x)}{(x - \mu_k)^2} \right\}_{k=1}^{K}, \quad (275)\]

where \(x \in \mathbb{R}\), is a Chebyshev system.

**Proof.** Let \(\varphi_1, \varphi_2, \ldots, \varphi_L: \mathbb{R} \to \mathbb{R}\) be defined by

\[\varphi_i(x) = \begin{cases} 
\frac{p(x)}{(x - \mu_i)^2} & \text{if } 1 \leq i \leq K, \\
\frac{p(x)}{x - \mu_i} & \text{if } K + 1 \leq i \leq L, 
\end{cases} \quad (276)\]
where $x \in \mathbb{R}$ and $p$ is defined by (274), for all integers $1 \leq i \leq L$. Furthermore, let $\psi_1, \psi_2, \ldots, \psi_K : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$
\psi_i(x) = \frac{p(x)}{x - \mu_i},
$$

(277)

where $x \in \mathbb{R}$ and $p$ is defined by (274), for all integers $1 \leq i \leq K$. Suppose that

$$
C_i = \prod_{\substack{\ell=1 \atop \ell \neq i}}^{L} (\mu_i - \mu_\ell) \prod_{k=1 \atop k \neq i}^{K} (\mu_i - \mu_k),
$$

(278)

for all integers $1 \leq i \leq L$. Then it is straightforward to show that

$$
\frac{\varphi_i(x_j)}{C_i} = \begin{cases} 1 & j = i, \\ 0 & j \neq i, \end{cases}
$$

(279)

for all integers $1 \leq i, j \leq L$, and

$$
\psi_i(\mu_j) = 0,
$$

(280)

for all integers $1 \leq i \leq K$, $1 \leq j \leq L$, and

$$
\frac{\psi_i'(x_j)}{C_i} = \begin{cases} 1 & j = i, \\ 0 & j \neq i, \end{cases}
$$

(281)

for all integers $1 \leq i, j \leq K$.

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$
f(x) = \sum_{i=1}^{L} b_i \frac{\varphi_i(x)}{C_i} + \sum_{i=1}^{K} c_i \frac{\psi_i(x)}{C_i},
$$

(282)

where $x \in \mathbb{R}$ and $C_i$ is defined by (278), and where $b_1, b_2, \ldots, b_L$ and $c_1, c_2, \ldots, c_K$ are arbitrary real numbers. Suppose further that there exists a set of $L + K$ distinct points $y_1, y_2, \ldots, y_{L+K}$ such that

$$
f(y_i) = 0,
$$

(283)

for each integer $1 \leq i \leq L + K$. Since $f$ is a polynomial of order at most $L + K - 1$ with $L + K$ distinct roots, it must be uniformly zero. We will show that this implies $b_i = 0$ for all $1 \leq i \leq L$ and $c_i = 0$ for all $1 \leq i \leq K$.

First, suppose that $b_i \neq 0$ for some integer $1 \leq i \leq L$. Then, by (282), (279), and (280),

$$
f(\mu_i) = b_i,
$$

(284)

so $f$ is not uniformly zero.

Next, suppose that $c_i \neq 0$ for some integer $1 \leq i \leq K$. If $b_\ell \neq 0$ for some $1 \leq \ell \leq L$, then $f$ is not uniformly zero, so suppose further that $b_\ell = 0$ for all $1 \leq \ell \leq L$. Combining (282) with (281), we have

$$
f'(\mu_i) = c_i,
$$

(285)

so $f$ is not uniformly zero.
Therefore, the only vector \((b_1, b_2, \ldots, b_L, c_1, c_2, \ldots, c_K)^T\) for which

\[
\begin{pmatrix}
\varphi_1(y_1) & \ldots & \varphi_L(y_1) & \psi_1(y_1) & \ldots & \psi_K(y_1) \\
\varphi_1(y_2) & \ldots & \varphi_L(y_2) & \psi_1(y_2) & \ldots & \psi_K(y_2) \\
\vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
\varphi_1(y_{L+K}) & \ldots & \varphi_L(y_{L+K}) & \psi_1(y_{L+K}) & \ldots & \psi_K(y_{L+K})
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_L \\
c_1 \\
c_2 \\
\vdots \\
c_K
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\] (286)

is true is the zero vector. Thus

\[
\det
\begin{pmatrix}
\varphi_1(y_1) & \ldots & \varphi_L(y_1) & \psi_1(y_1) & \ldots & \psi_K(y_1) \\
\varphi_1(y_2) & \ldots & \varphi_L(y_2) & \psi_1(y_2) & \ldots & \psi_K(y_2) \\
\vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
\varphi_1(y_{L+K}) & \ldots & \varphi_L(y_{L+K}) & \psi_1(y_{L+K}) & \ldots & \psi_K(y_{L+K})
\end{pmatrix} \neq 0,
\] (287)

for any set of \(L + K\) distinct points \(y_1, y_2, \ldots, y_{L+K}\). \(\Box\)

References

