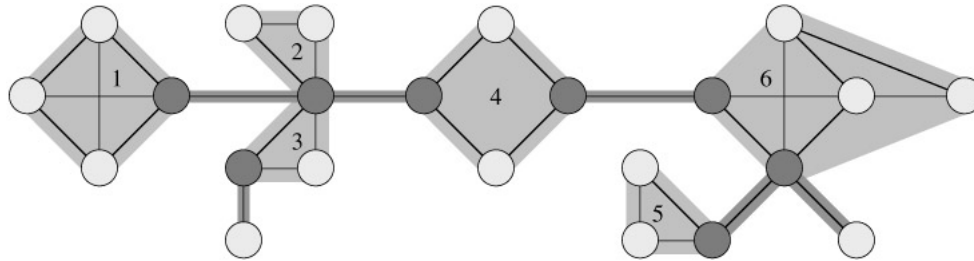


Homework Assignment #4
Due: December 6, 2007, by 10:10 am

1. Please complete and attach (with a staple) an assignment cover page to the front of your assignment. You may work alone or with one other student. If you work in a group, write both your names on the cover sheet and submit only one copy of your homework.
2. If you do not know the answer to a question, and you write “I (We) do not know the answer to this question”, you will receive 20% of the marks of that question. If you just leave a question blank with no such statement, you get 0 marks for that question.
3. Unless we explicitly state otherwise, you should justify your answers. Your paper will be marked based on the correctness and completeness of your answers, and the clarity, precision and conciseness of your presentation.

Question 1. (35 marks) Let $G = (V, E)$ be a connected undirected graph. Let $n = |V|$ and $m = |E|$. An *articulation point* of G is a vertex whose removal disconnects G . A *bridge* of G is an edge whose removal disconnects G . A *biconnected component* of G is a maximal set of edges such that any two edges in the set lie on a common simple cycle. The following diagram illustrates these definitions: the articulation points are the dark vertices, the bridges are the dark shaded edges, and the edges of the biconnected components are in the shaded regions with a *bcc* number. We can determine articulation points, bridges, and biconnected components using depth-first search. Let $T = (V, E_T)$ be a depth-first search tree of G .



- a. (6 marks) Prove that the root of T is an articulation point of G if and only if it has at least two children in T . (Remember to prove both directions of the iff!)
- b. (5 marks) Let v be a non-root vertex in T . Prove that v is an articulation point of G if and only if, in T , v has a child s such that there is no back edge (u, w) such that u is a descendant of s and w is a proper ancestor of v . (Note that a vertex is its own descendant and ancestor, but not a proper descendant nor proper ancestor.)
- c. (5 marks) Let $low[u] = \min(\{d[u]\} \cup \{d[w] \mid (v, w) \text{ is a back edge from a descendant } v \text{ of } u\})$. Show how to compute $low[u]$ for all vertices $u \in V$ in $O(m)$ time.
- d. (2 marks) Show how to compute all articulation points in $O(m)$ time.
- e. (4 marks) Prove that an edge of G is a bridge if and only if it is not part of any simple cycle of G .
- f. (5 marks) Show how to compute all the bridges of G in $O(m)$ time.
- g. (3 marks) Prove that the biconnected components of G partition the non-bridge edges of G .
- h. (5 marks) Give an algorithm that in $O(m)$ time labels each edge e of G with a positive integer $bcc[e]$ such that $bcc[e] = bcc[e']$ if and only if e and e' are in the same biconnected component.

Question 2. (15 marks) Let $G = (V, E)$ be a connected undirected graph where each edge $e \in E$ has positive integer weight $w(e)$.

a. (5 marks) Suppose no two edges of G have the same weight. Prove that G has exactly one minimum-weight spanning tree.

b. (5 marks) Suppose all edges of G have unique weight except for the two edges with lowest weight, which both have the same weight. Prove or disprove that G has exactly one minimum-weight spanning tree.

c. (5 marks) Suppose all edges of G have unique weight except for the three edges with lowest weight, all three of which have the same weight. Construct two weighted graphs G_1 and G_2 , each containing at least six edges, such that G_1 has exactly *one* MST, and G_2 has at least *two* different MSTs. Briefly justify that your graphs satisfy the stated requirements.

Question 3. (40 marks) This question is a programming assignment. To see its description follow the link given in the “Assignments” section of the course web page.