Duration: **50 minutes**
Aids Allowed: **NONE** (in particular, no calculator)

Student Number: |_|_|_|_|_|_|_|_|_|

Last (Family) Name(s): SOLUTIONS

First (Given) Name(s): SAMPLE

Tutorial Section: (circle one)	BA 2165 Steve	BA 2159 Alex	BA 2175 Anatoliy

*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below *carefully*.)

This term test consists of 3 questions on 5 pages (including this one), printed on one side of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete and write your student number where indicated at the bottom of every page* (except page 1).

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

Remember that, when we ask for a proof using our structured proof form, you may omit the conclusion lines and combine basic steps as long as the structure of your proof is clear. Also, unless otherwise stated, you do not need to include justifications for steps that a reader would readily believe.

Good Luck!

Marking Guide

1: ______/12

2: ______/16

3: ______/12

TOTAL: ______/40

Question 1. [12 MARKS]

Consider the following beginning to a proof:

Assume $x \in D$
  $Q(x)$
  Assume $P(x)$
    Let $y = f(x)$. Then $y \in D$.
    $A(x, y)$
    $\neg Q(y)$

Which of the following statements may we legitimately conclude following the above argument? For each statement that *may* legitimately be concluded, give the rule(s) of inference that allows us to conclude the statement. For each statement that may *not* be legitimately concluded, give a short justification explaining why the statement may be false.

(i) $\forall z \in D, \neg Q(z)$ Circle one choice: valid conclusion invalid conclusion

Justification:

Invalid: it's entirely possible that there's a $z \in D$ having $Q(z)$ true (x would be such an element whenever D is not empty)

(ii) $\exists v \in D, A(x, v)$ Circle one choice: valid conclusion invalid conclusion

Justification:

Valid: by ∃I, we use y to show that $\exists y \in D, A(x, y)$; this statement is just a renaming of the quantified variable

(iii) $P(x) \Rightarrow Q(x)$ Circle one choice: valid conclusion invalid conclusion

Justification:

Valid: under the assumption of $P(x)$, we know $Q(x)$ is true (this was concluded earlier, but we could also rewrite it inside the assumptions' scope), so ⇒I allows us to conclude this statement

(iv) $\exists w \in D, P(w)$ Circle one choice: valid conclusion invalid conclusion

Justification:

Invalid: we don't know that $P()$ is true about any element in D, only some facts in the case when $P(x)$ happens to be true (we don't even know for sure that there *is anything* in D yet!)

MARKING SCHEME:

- 3 points each part, for reasonable-ness of justification and degree that given justification supports choice of valid/invalid (generally, 2 points for justification, 1 point for valid/invalid)

Question 2. [16 MARKS]

Consider the following statement about sequences of natural numbers:

(S) $(\forall i \in \mathbb{N}, a_i \geq i \wedge \forall j \in \mathbb{N}, a_j \leq j^2) \Rightarrow (\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, a_{m+1} \leq a_k + a_m)$

Part (a) [7 MARKS]

Write a sequence of natural numbers for which (S) is true. Then, using our structured proof format, prove that (S) is true for your given sequence.

SAMPLE SOLUTION

To satisfy (S), we can either make sure both parts of the implication are true, or make sure the first part (the antecedent) is false. The second method is probably easier.

Thus, consider the sequence given by $a_n = 0$. We use the following proof, rewriting the implication as a disjunction.

Let $i = 2$. Then $i \in \mathbb{N}$.
Then $a_i = 0 < i = 2$.
So $\exists i \in \mathbb{N}, a_i < i$. (by ∃I)
Then $\exists i \in \mathbb{N}, a_i < i \vee \neg\forall j \in \mathbb{N}, a_j \leq j^2$. (by ∨I)
Then $\neg(\forall i \in \mathbb{N}, a_i \geq i \wedge \forall j \in \mathbb{N}, a_j \leq j^2)$. (by DeMorgan's law)
Then $\neg(\forall i \in \mathbb{N}, a_i \geq i \wedge \forall j \in \mathbb{N}, a_j \leq j^2) \vee (\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, a_{m+1} \leq a_k + a_m)$. (by ∨I)
Thus $(\forall i \in \mathbb{N}, a_i \geq i \wedge \forall j \in \mathbb{N}, a_j \leq j^2) \Rightarrow (\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, a_{m+1} \leq a_k + a_m)$. (by rewriting as an implication)

(Another easy solution is to use the sequence $a_n = n$ and prove the implication directly.)

MARKING SCHEME:

A. 1 mark: correctness of sequence
B. 3 marks: appropriateness of proof structure
C. 3 marks: correctness of argument
- up to −2 marks for omitting necessary justification

Part (b) [9 MARKS]

Recall that statement (S) says:

(S) $(\forall i \in \mathbb{N}, a_i \geq i \wedge \forall j \in \mathbb{N}, a_j \leq j^2) \Rightarrow (\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, a_{m+1} \leq a_k + a_m)$

Write a sequence of natural numbers for which (S) is false. Then, using our structured proof format, prove that (S) is false for your given sequence.

SAMPLE SOLUTION

The negation of (S) is $(\forall i \in \mathbb{N}, a_i \geq i \wedge \forall j \in \mathbb{N}, a_j \leq j^2) \wedge (\forall k \in \mathbb{N}, \exists m \in \mathbb{N}, a_{m+1} > a_k + a_m)$.

So we need a sequence where there's always a value larger than the sum of the previous value and some (smaller) value. Let's consider the fastest growing sequence that satisfies the antecedent: $a_n = n^2$. We note that $a_{m+1} = (m+1)^2 = m^2 + 2m + 1$, so the "smaller value" needs to be no more than $2m$. We simply need to pick m big enough so that this happens.

The proof is as follows:

Assume $i \in \mathbb{N}$.
  Then $a_i = i^2 \geq i$.
  And $a_i = i^2 \leq i^2$.
Thus $\forall i \in \mathbb{N}, a_i \geq i$ (by $\forall$I) and $\forall j \in \mathbb{N}, a_j \leq j^2)$ (by $\forall$I).
Assume $k \in \mathbb{N}$.
  Let $m = k^2$. Then $m \in \mathbb{N}$.
  Then $a_k = k^2 = m < 2m + 1$.
  So $a_{m+1} = (m+1)^2 = m^2 + 2m + 1 = a_m + (2m + 1) > a_m + a_k$.
  Thus $\exists m \in \mathbb{N}, a_{m+1} > a_k + a_m$. (by $\exists$I)
Thus $\forall k \in \mathbb{N}, \exists m \in \mathbb{N}, a_{m+1} > a_k + a_m$. (by $\forall$I)
Hence $(\forall i \in \mathbb{N}, a_i \geq i \wedge \forall j \in \mathbb{N}, a_j \leq j^2) \wedge (\forall k \in \mathbb{N}, \exists m \in \mathbb{N}, a_{m+1} > a_k + a_m)$. (by $\wedge$I)

MARKING SCHEME:

A. 2 marks: writing (or implicitly showing) negation of (S)
B. 1 mark: correctness of sequence
C. 3 marks: appropriateness of proof structure
D. 3 marks: correctness of argument

- up to -2 marks for omitting necessary justification

Question 3. [12 marks]

Recall that $g \in O(f)$ iff $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.

Using our structured proof format, prove or disprove that $g \in O(f)$,

where $g(n) = \begin{cases} 4n^2 - 3n + 6, & \text{if } n \text{ is even} \\ 100, & \text{if } n \text{ is odd} \end{cases}$ and $f(n) = \begin{cases} n^2 - 4, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd} \end{cases}$

Sample Solution

Let $c = 50$. Then $c \in \mathbb{R}^+$.

Let $B = 4c = 200$. Then $B \in \mathbb{N}$ (since also $c \in \mathbb{N}$).

Assume $n \in \mathbb{N}$.
  Assume $n \geq B$.
    Then n is either odd or even.
    Case 1: Assume n is odd.
      Then $g(n) = 100 = 50 \cdot 2 = c \cdot 2 \leq c(2n) = cf(n)$. (since $n \geq 200$)
    Case 2: Assume n is even.
      Then $g(n) = 4n^2 - 3n + 6 \leq 4n^2 + 6 \leq 4n^2 + 6n^2 = 10n^2$.
      Also, $10n^2 = 11n^2 - n^2 \leq 11n^2 - n \leq cn^2 - 4c$ since $c \geq 11$ and $n \geq 4c$.
      Thus $g(n) \leq cn^2 - 4c = c(n^2 - 4) = cf(n)$.
    Since we concluded it in each case, $g(n) \leq cf(n)$.
  Thus $n \geq B \Rightarrow g(n) \leq cf(n)$.

Since n an arbitrary member of $\mathbb{N}$, then $\forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.

Since $B \in \mathbb{N}$, then $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.

Thus $g \in O(f)$ (by definition of $O(f)$).

Marking Scheme:

A. 2 marks: recognizing it should be a proof
B. 3 marks: general structure of proof
C. 1 mark: cases on n being even or odd
D. 2 marks: appropriateness of choice of c, B
E. 1 mark: correctness of argument for odd case
F. 3 marks: correctness of argument for even case and handling negative terms correctly

- up to -3 marks for omitting necessary justification