University of Toronto, St. George Campus — CSC 165H1 F

Department of Computer Science — Fall 2007

Quiz #8 Sample Solutions

We want to prove the following statement:

$$2n\sqrt{n} \;\notin\; O(n+4)$$

We first write what we need to prove (the negation of the property for big-O):

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2n\sqrt{n} > c(n+4)$$

We construct the standard proof structure:

Assume $c \in \mathbb{R}^+$.
  Assume $B \in \mathbb{N}$.
    Let $n =$ ____.
    Then $n \in \mathbb{N}$.
    Thus $n \geq B$.
    $\vdots$
    Thus $2n\sqrt{n} > c(n+4)$.
    So $n \geq B \wedge 2n\sqrt{n} > c(n+4)$. (by $\wedge$I)
    Thus $\exists n \in \mathbb{N}, n \geq B \wedge 2n\sqrt{n} > c(n+4)$. (by $\exists$I)
  Since B is an arbitrary element of $\mathbb{N}$, $\forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2n\sqrt{n} > c(n+4)$. (by $\forall$I)
Since c is an arbitrary element of $\mathbb{R}^+$, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2n\sqrt{n} > c(n+4)$. (by $\forall$I)
By definition, $2n\sqrt{n} \;\notin\; O(n+4)$.

We need to find a value for n that will work. We do some scratch work:

$$\begin{aligned}
\text{To show: } \quad 2n\sqrt{n} &> c(n+4) \\
\text{We need: } \quad 2n\sqrt{n} &> c(n+n) = 2cn \quad \text{(need } n \geq 4\text{)} \\
\text{So: } \quad \sqrt{n} &> c \quad \text{(divide both sides by } 2n\text{)} \\
n &> c^2 \quad \text{(square both sides)}
\end{aligned}$$

So we must pick n such that $n \geq 4$ and $n > c^2$ (also, we need $n \geq B$ for the proof structure). Picking $n = \max\{c^2+1, B, 4\}$ will do. We turn our scratch work upside-down and turn it into a proof (remember, we only want to write true things in a proof!).

We fill in the $\vdots$ with:

Then $c > c^2$. (since $c \geq c^2 + 1$)
So $\sqrt{n} > c$. (taking square root of each side)
Then $2n\sqrt{n} > 2n \cdot c$. (multiplying both sides by $2n$)
Now $2n \cdot c = c(n+n) \geq c(n+4)$. (since $n \geq 4$)
Thus $2n\sqrt{n} > c(n+4)$. (by transitivity of $>$)

This completes the proof.