University of Toronto, St. George Campus — CSC 165H1 F
Department of Computer Science — Fall 2006

Quiz #6 Sample Solutions

We want to prove the following statement:

$$[(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} > 5a_i)] \Rightarrow (\exists i \in \mathbb{N}, a_i > 100)$$

The first thing we should do is determine a valid structure for the proof.

Assume $(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} > 5a_i)$.
  Then $a_1 \geq 1$ (by $\wedge$E).
  And $\forall i \in \mathbb{N}, a_{2i} > 5a_i$ (by $\wedge$E).
  Let $i =$ ____.
    Then $i \in \mathbb{N}$.
    $\vdots$
    Thus $a_i > 100$ (by ____).
  Thus $\exists i \in \mathbb{N}, a_i > 100$ (by $\exists$I).
Therefore, $[(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} > 5a_i)] \Rightarrow (\exists i \in \mathbb{N}, a_i > 100)$ (by $\Rightarrow$I).

We're most of the way there now. We just need to pick a value for i that will let us prove $a_i > 100$. Now is the time for a little thought. What value will work? If we're clever, we can probably figure it out. Or we can see what that universal statement will tell us. We already know something about a_1, so wouldn't it be nice to learn something about a_2? That universal will work nicely!

  $1 \in \mathbb{N}$ (by definition of natural numbers).
  $a_2 > 5a_1$ (by $\forall$E for $i = 1$).
  So $a_2 > 5 \cdot 1 = 5$ (by transitivity of $>$).

We can go on for 2 to learn about a_4, then about a_8:

  $2 \in \mathbb{N}$ (by definition of natural numbers).
  $a_4 > 5a_2$ (by $\forall$E for $i = 2$).
  So $a_4 > 5 \cdot 5 = 25$ (by transitivity of $>$).
  $4 \in \mathbb{N}$ (by definition of natural numbers).
  $a_8 > 5a_4$ (by $\forall$E for $i = 4$).
  So $a_8 > 5 \cdot 25 = 125$ (by transitivity of $>$).

So I guess we should pick $i = 8$ in our proof. Here's the completed version:

Assume $(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} > 5a_i)$.
  Then $a_1 \geq 1$ (by $\wedge$E).
  And $\forall i \in \mathbb{N}, a_{2i} > 5a_i$ (by $\wedge$E).
  Since $1 \in \mathbb{N}$, $a_2 > 5a_1$ (by $\forall$E). So $a_2 > 5 \cdot 1 = 5$ (by transitivity of $>$).
  Since $2 \in \mathbb{N}$, $a_4 > 5a_2$ (by $\forall$E). So $a_4 > 5 \cdot 5 = 25$ (by transitivity of $>$).
  Since $4 \in \mathbb{N}$, $a_8 > 5a_4$ (by $\forall$E). So $a_8 > 5 \cdot 25 = 125$ (by transitivity of $>$).
  Let $i = 8$.
    Then $i \in \mathbb{N}$.
    Since $125 > 100$ (by math), we get $a_i > 100$ (by transitivity of $>$).
  Thus $\exists i \in \mathbb{N}, a_i > 100$ (by $\exists$I).
Therefore, $[(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} > 5a_i)] \Rightarrow (\exists i \in \mathbb{N}, a_i > 100)$ (by $\Rightarrow$I).

Note that it's okay to put our reasoning about $a_2, \ldots, a_8$ in the $\vdots$ part of our original proof outline instead.