Quiz #5 Sample Solutions

We must show the proof structure we would use to prove:

(S) $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$

The outermost logical symbol is a universal quantifier, so we'll need to show the inside (the existentially quantified statement) is true for an arbitrary natural number:

Assume $i \in \mathbb{N}$.
  ...
  Then $\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.
Since this is true for an arbitrary natural number, then $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.

We still need to prove the existentially quantified sub-statement, so we'll need to find an example (and prove that it satisfies the statement). Filling this in gives us:

Assume $i \in \mathbb{N}$.
  Let $k = \_\_$.
  Then $k \in \mathbb{N}$.
  ...
  Then $\forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.
  Since this is true for a particular natural number, then $\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.
Since this is true for an arbitrary natural number, then $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.

We still need to show a universal statement, so we need to prove the inside about an arbitrary natural number. So:

Assume $i \in \mathbb{N}$.
  Let $k = \_\_$.
  Then $k \in \mathbb{N}$.
  Assume $m \in \mathbb{N}$.
    ...
    Then $m \leq k \Rightarrow a_m \leq {a_i}^2$.
  Since this is true for an arbitrary natural number, then $\forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.
  Since this is true for a particular natural number, then $\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.
Since this is true for an arbitrary natural number, then $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.

All that's left is to prove an implication, so we'll put in the direct proof of implication structure:

Assume $i \in \mathbb{N}$.
  Let $k = \_\_$.
  Then $k \in \mathbb{N}$.
  Assume $m \in \mathbb{N}$.
    Assume $m \leq k$.
      ...
      Then $a_m \leq {a_i}^2$.
    Then $m \leq k \Rightarrow a_m \leq {a_i}^2$.
  Since this is true for an arbitrary natural number, then $\forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.
  Since this is true for a particular natural number, then $\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.
Since this is true for an arbitrary natural number, then $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq k \Rightarrow a_m \leq {a_i}^2$.