Quiz #4 Sample Solutions

Consider the following two statements:

(1) $A(x) \Rightarrow (B(x) \Rightarrow C(x))$

(2) $(A(x) \wedge B(x)) \Rightarrow C(x)$

a. Using any method you want (Venn diagrams, manipulating sentences, logical arithmetic, truth table, direct proof, etc.), prove that (1) implies (2).

There are many different ways to prove this statement.

Venn diagrams: We draw two Venn diagrams (one for each statement) each with three intersecting circles A, B and C, and shade the regions where the statement is true. For (1), anything outside A is good, so we shade outside A. Inside A, we have $B(x) \Rightarrow C(x)$, so we shade everything outside B or inside C. For (2), anything outside $A \cap B$ is good, so we shade everything outside this intersection. Inside the intersection, only things in C are good, so we shade inside C. Finally, we compare the diagrams and see that every shaded region in (1) is shaded in (2), so the statement is proven. (To use this technique, you'll need to justify why each region is shaded.)

Manipulating sentences / using logical arithmetic: We start with (1), and using our rules, try to get (2). Start with (1): $A(x) \Rightarrow (B(x) \Rightarrow C(x))$
$\neg A(x) \vee (B(x) \Rightarrow C(x))$ by converting the first implication to a disjunction.
$\neg A(x) \vee (\neg B(x) \vee C(x))$ by converting the remaining implication to a disjunction.
$(\neg A(x) \vee \neg B(x)) \vee C(x)$ by associativity of $\vee$.
$\neg(A(x) \wedge B(x)) \vee C(x)$ by De Morgan's Law.
$(A(x) \wedge B(x)) \Rightarrow C(x)$ by conversion of disjunction to implication, which is precisely (2).

Truth table: We write out the truth tables for each formula and compare the columns.

$A(x)$	$B(x)$	$C(x)$	$B(x) \Rightarrow C(x)$	(1)	$A(x) \wedge B(x)$	(2)	(1)$\Rightarrow$(2)
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

Whenever (1) is true, (2) is true (alternatively, the rightmost column is always true), so the statement is proven.

Direct proof: We mimic the structure we saw in lecture.
(i) Given x such that $A(x) \wedge B(x)$ is true,
  (ii) $A(x)$ from line (i) and conjunction elimination
  (iii) $B(x) \Rightarrow C(x)$ from (1) and (ii)
  (iv) $C(x)$ from (i) and (iii)
(v) Hence $(A(x) \wedge B(x)) \Rightarrow C(x)$ by the subargument (i)-(iv)

Any of these methods (and some others) are acceptable. You might want to try proving (b) yourself using each of these methods.

b. Using any method you want, prove that (2) implies (1).

Any of the methods we used for (a) are acceptable here too. In fact, we realize that for many of the methods, the work we did in (a) helps us prove (b) quite a bit.

Venn diagrams: Using the same diagrams from (a), we see that every region shaded in (2) is also shaded in (1), so the statement is proven.

Manipulating sentences / using logical arithmetic: We just do the same thing in reverse (since all our manipulation steps go either direction.

Truth table: We notice that whenever (2) is true, (1) is true. Or equivalently, we work out that (2) $\Rightarrow$ (1) is always true.

Direct proof: We use the same basic structure, only this time to show (1) using (2).

- (i) Given x such that $A(x)$ is true,
  - (ii) Assume $B(x)$ is true.
    - (iii) Then $A(x) \wedge B(x)$ from (i) and (ii)
    - (iv) $C(x)$ from (2) and (iii)
  - (v) Hence $B(x) \Rightarrow C(x)$ by the subargument (ii)-(iv) (under the assumption of (i)!)
- (vi) Hence $A(x) \Rightarrow (B(x) \Rightarrow C(x))$ by the subargument (i)-(v)

Pay close attention to the structure of this argument (which we used twice here): to prove $P \Rightarrow Q$, we assume P and derive Q.

c. Write a statement using $\Leftrightarrow$ that summarizes the conclusions of (a) and (b).

A fully correct answer would be something like:

$$(A(x) \Rightarrow (B(x) \Rightarrow C(x))) \iff ((A(x) \wedge B(x)) \Rightarrow C(x))$$

Note the parentheses to enforce the correct order of connectives. If would also be correct to universally quantify this statement:

$$\forall x \in D, (A(x) \Rightarrow (B(x) \Rightarrow C(x))) \iff ((A(x) \wedge B(x)) \Rightarrow C(x))$$

but we'd have to define a domain D.

Answers such as (1) $\Leftrightarrow$ (2) are also acceptable.