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UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2007 EXAMINATIONS

CSC 165H1F
Instructor: R. Krueger

Duration — 3 hours

No Aids Allowed

Student Number: ____________

Last (Family) Name(s): ____________

First (Given) Name(s): ____________

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below *carefully*.)

This final examination consists of 10 questions on 14 pages (including this one), printed on one side of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete and write your student number where indicated at the bottom of every page* (except page 1).

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

When giving a symbolic statement, if there is more than one interpretation, be sure to use parentheses to make your answer clear.

Where a proof is required, if your proof is well-structured and clear, then the standard ending summaries for quantification and implication are not required. You will earn substantial part marks for writing the outline of the proof and indicating which steps are missing, even if you cannot complete the proof.

Good Luck!

MARKING GUIDE

# 1:	____/ 10
# 2:	____/ 8
# 3:	____/ 6
# 4:	____/ 8
# 5:	____/ 15
# 6:	____/ 5
# 7:	____/ 10
# 8:	____/ 20
# 9:	____/ 14
# 10:	____/ 4
TOTAL:	____/100

Question 1. [10 MARKS]

The students in the new Intro to Game Design capstone course recently showed off the video games they developed. They were all quite impressive, and some of the following statements were made by the audience. We define the following domains and predicates:

G represents the set of video games that were demonstrated
C represents the set of companies that publish video games
$good(x)$ represents "game x looks good"
$fun(x)$ represents "game x is fun"
$crash(x)$ represents "game x crashed during the demonstration"
$interested(c, x)$ represents "video game company c is interested in game x"

Write each of the following statements symbolically using the above domains and predicates.

(a) Every one of the games look good.

(b) Some of the games are fun.

(c) At least two of the games crashed.

(d) The games that crashed weren't all that fun.

(e) A video game company is interested in publishing some, but not all, of the games that didn't crash.

Question 2. [8 MARKS]

Three tourists visited Toronto and each bought two souvenir items:

Xia bought a mini-CN tower and a postcard.
Yulia bought a postcard and a T-shirt.
Zach bought some maple syrup and a hat.

Let $T = \{\text{Xia, Yulia, Zach}\}$ and let $S = \{\text{hat, mini-tower, postcard, syrup, T-shirt}\}$.
Let $bought(t, i)$ represent "person t bought item i."

Part (a) [1 MARK]

Write the following statement in everyday English: $\exists i \in S, \forall t \in T, bought(t, i)$

Part (b) [1 MARK]

Is the sentence in (a) true or false?

Part (c) [1 MARK]

Write the following statement in everyday English: $\forall t \in T, \exists i \in S, bought(t, i)$

Part (d) [1 MARK]

Is the sentence in (c) true or false?

Part (e) [4 MARKS]

Write a single sentence which is open in t that is

- false for $t =$ Xia,
- false for $t =$ Yulia,
- true for $t =$ Zach,

and does not use any constants (i.e., does not mention Xia, Yulia, Zach, hat, mini-tower, postcard, syrup, or T-shirt).

Question 3. [6 MARKS]

One of the proof structures we saw in the lectures was the following proof by cases:

> $A \vee B$
> Case 1: Assume A
> $\quad\vdots$
> $\quad$Then C
> Case 2: Assume B
> $\quad\vdots$
> $\quad$Then C
> Since $A \vee B$ and in both cases we concluded C, then C.

One of your fellow students pointed out that the structure used inside each case looks more like a direct proof of an implication. For example, Case 1 allows us to conclude $A \Rightarrow C$.

Part (a) [5 MARKS]

This proof structure actually uses the following identity: $[(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)] \Rightarrow C$.
Using any method you like, prove this identity.

Part (b) [1 MARK]

Using this insight, name the basic rule of inference by which we conclude C in the above proof structure.

Question 4. [8 MARKS]

Consider the following start to a proof (the justifications and definitions are omitted; assume everything is correct):

Assume $x \in \mathbb{N}$.
  Let $y = 165x$. Then $y \in \mathbb{N}$.
  $A(y)$.
  Assume $B(x)$.
    $\neg B(y)$.
    $C(y, x) \vee \neg C(x, y)$.
    $\neg D(x)$.
    line 1
  line 2
line 3

For each of the following statements, indicate whether we may conclude it using only the facts and arguments already mentioned in the proof at lines 1, 2 or 3 (circle *all* the locations we may conclude it) or whether we cannot conclude the statement at any of these locations. Briefly justify your choice(s).

(i) $\exists n \in \mathbb{N}, A(n)$ line 1 line 2 line 3 none of these

Justification:

(ii) $C(x, 165x)$ line 1 line 2 line 3 none of these

Justification:

(iii) $\forall i \in \mathbb{N}, B(i) \Rightarrow \neg D(i)$ line 1 line 2 line 3 none of these

Justification:

(iv) $\neg\exists k \in \mathbb{N}, \forall m \in \mathbb{N}, (B(k) \wedge B(m))$ line 1 line 2 line 3 none of these

Justification:

Question 5. [15 MARKS]

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Consider the following statement about infinite sequences of natural numbers $a_0, a_1, a_2, \ldots$:

$$\text{(S)} \quad \exists i \in \mathbb{N}, i \geq 1 \wedge \forall k \in \mathbb{N}, a_{i+k} = a_k$$

Part (a) [5 MARKS]

Using our structured proof form, prove that (S) is *true* for the sequence $1, 2, 1, 2, 1, 2, \ldots$.

You may use the fact that $a_i = \begin{cases} 1, & \text{if } i \text{ is even} \\ 2, & \text{if } i \text{ is odd} \end{cases}$.

Part (b) [10 MARKS]

Recall that statement (S) said:

$$\text{(S)} \quad \exists i \in \mathbb{N}, i \geq 1 \wedge \forall k \in \mathbb{N}, a_{i+k} = a_k$$

Using our structured proof form, prove that (S) is *false* for the sequence $1, 1, 2, 1, 2, 1, 2, \ldots$.

You may use the fact that $a_i = \begin{cases} 1, & \text{if } i \text{ is odd or } i = 0 \\ 2, & \text{if } i \text{ is even and } i \neq 0 \end{cases}$.

Question 6. [5 MARKS]

Recall that for $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, we say $g \in O(f)$ iff $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)$.
Remember that you may omit the ending summaries if your proof is well-structured and clear.

For any function $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, define $\bar{g} : \mathbb{N} \to \mathbb{R}^{\geq 0}$ by $\bar{g}(n) = \lceil g(n) \rceil$.
Using our structured proof form, prove or disprove that for all $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f \in O(g)$, then $f \in O(\bar{g})$.

Question 7. [10 MARKS]

Let $\mathcal{F} = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0}\}$.
Recall that, for $f \in \mathcal{F}$, $O(f) = \{g \in \mathcal{F} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n)\}$.

For any function $f \in \mathcal{F}$, define $\Gamma(f) = \{g \in \mathcal{F} \mid \exists a \in \mathbb{R}^+, \exists d \in \mathbb{R}, \forall n \in \mathbb{N}, g(n) \leq af(n) + d\}$.
Using our structured proof form, prove that $\forall f \in \mathcal{F}, g \in O(f) \Rightarrow g \in \Gamma(f)$.

Question 8. [20 marks]

Consider the following mystery method (assume $n > 0$):

```
int mystery(int n) {   // Precondition: n > 0
   int x = n;
   int y = 0;
   int p = 1;
   int d = 0;
   while ( x > 1 ) {   // Invariant: x * 2^y == n - d and p == 2^y
      if ( odd(x) ) {  // odd(x) returns true if x % 2 == 1
         x = x - 1;
         d = d + p;
      } else  {
         x = x / 2;
         y = y + 1;
         p = p * 2;
      }
   }
   return y;
}
```

Part (a) [10 marks]

The author claims that $x \cdot 2^y = n - d \land p = 2^y$ is a loop invariant for this method. Prove that if the claimed loop invariant is true at the *beginning* of one loop iteration, then it is true at the *end* of that one loop iteration.

Part (b) [10 MARKS]

Prove that the worst-case number lines executed by the `mystery(n)` method is in $\Theta(\log_2 n)$.
(Recall that $\log_b n = x$ iff $b^x = n$.)

Question 9. [14 MARKS]

Consider the normalized floating point system $\mathcal{F}$ with $\beta = 2$, $t = 4$, $e_{\max} = 3$ and $e_{\min} = -3$.
Recall that $\frac{1}{8} = 0.125$ and $\frac{1}{16} = 0.0625$.

Part (a) [2 MARKS]

How many positive (and non-zero) numbers are there in $\mathcal{F}$? (You do not need to simplify your answer.)

Part (b) [2 MARKS]

Give the smallest and largest positive (and non-zero) numbers representable in $\mathcal{F}$. Explain your answer.

Part (c) [2 MARKS]

Give all the real numbers that can be represented exactly in $\mathcal{F}$ within the real interval $[1\frac{3}{4}, 2]$.
(You may leave your answers as fractions.)

Part (d) [4 MARKS]

Represent 1.1 and 1.32 (both currently written in base 10) in the system $\mathcal{F}$ by rounding to the *nearest* number in the system.

Part (e) [4 MARKS]

Give the relative error for each representation in (iv), expressed as a fraction in lowest terms.

Question 10. [4 MARKS]

Part (a) [2 MARKS]

What is the condition number of f, where $f(x) = 2 - x$?

Part (b) [2 MARKS]

How is your answer related to catastrophic cancellation? Explain.

Total Marks = 100

(use this page for rough work or any answers that didn't fit)