Duration: **50 minutes**
Aids Allowed: **NONE** (in particular, no calculator)

Student Number: |_|_|_|_|_|_|_|_|_|

Last (Family) Name(s): SOLUTIONS

First (Given) Name(s): SAMPLE

Tutorial Section: **(circle one)**	LM 157 Peter	SS 2105 Hania / Matthew	SS 2128 Costis

*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below *carefully*.)

This term test consists of 3 questions on 4 pages (including this one), printed on one side of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete and write your student number where indicated at the bottom of every page* (except page 1).

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

When giving a symbolic statement, if there is more than one interpretation, be sure to use parentheses to make your answer clear.

Where a proof is required, if your proof is well-structured and clear then the standard ending summaries for quantification and implication are not required. You will earn substantial part marks for writing the outline of the proof and indicating which steps are missing.

Marking Guide

1: ______/10

2: ______/10

3: ______/ 8

TOTAL: ______/28

Good Luck!

Question 1. [10 MARKS]

Part (a) [2 MARKS]

Consider a number b represented in binary as $b_k \dots b_1 b_0$.
Suppose we shift b two bits to the left and fill with 1s on the right (instead of the usual 0s).
What is the new value represented, in terms of the original number b?

$b' = b * 4 + 3$

Part (b) [4 MARKS]

Write the following decimal (base 10) numbers in binary (base 2):

13: $(1101)_2$

8.75: $(100.11)_2$

Part (c) [4 MARKS]

Suppose you need to prove $A \Rightarrow B$. Present two different proof structures that you could use to prove this statement.

SAMPLE SOLUTION

Direct proof:

Assume A
...
Then B
Thus $A \Rightarrow B$

Indirect proof (of contrapositive):

Assume $\neg B$
...
Then $\neg A$
Thus $\neg B \Rightarrow \neg A$
Thus $A \Rightarrow B$

Other valid proof structures exist.

Question 2. [10 marks]

Recall that $g \in \Omega(f)$ iff $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \geq cf(n)$.

Prove or disprove using our structured proof format: $\frac{n^2}{n+1} \in \Omega(n^2)$
Hint: Try using something like $\frac{1}{c}$.

Sample Solution

We disprove the statement. We use the hint to help us come up with a good choice for n.

Let $c \in \mathbb{R}^+$ be arbitrary.
  Let $B \in \mathbb{N}$ be arbitrary.
    Let $n = \max\{B, \lceil \frac{1}{c} \rceil + 1\}$. Then $n \in \mathbb{N}$.
      Then $n \geq B$.
      Also, $n > \frac{1}{c}$, so $n^2 > \frac{n}{c}$, so $n < cn^2$.
      Now $\frac{n^2}{n+1} < \frac{n^2}{n} = n < cn^2$.
      Hence $\frac{n^2}{n+1} < cn^2$.
    So $\exists n \in \mathbb{N}, n \geq B \wedge \frac{n^2}{n+1} < cn^2$.
  Thus $\forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge \frac{n^2}{n+1} < cn^2$.

Thus $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge \frac{n^2}{n+1} < c(n^2)$

Therefore $\frac{n^2}{n+1} \notin \Omega(n^2)$

Question 3. [8 MARKS]

Consider the following statement (from the previous test) about sequences of natural numbers $a_0, a_1, a_2, \ldots$ (recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$):

(T) $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, (a_i = a_k \Rightarrow a_i < k)$

Prove that (T) is true for all sequences. (*Hint:* Either $\forall j \in \mathbb{N}, a_i = a_j$ or not.)

SAMPLE SOLUTION

Observe that an implication can be satisfied if the first part is false or the second part is true.

Thus, we need only to pick a k such that $a_i < k$ (which is a little more trivial than I intended):

Let $i \in \mathbb{N}$ be arbitrary.
  Let $k = a_i + 1$. Then $k \in \mathbb{N}$.
    So $a_i < k$.
    Thus $a_i \neq a_k \vee a_i < k$, which means $a_i = a_k \Rightarrow a_i < k$
  Thus $\exists k \in \mathbb{N}, (a_i = a_k \Rightarrow a_i < k)$.
Since i is an arbitrary element of $\mathbb{N}$, $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, (a_i = a_k \Rightarrow a_i < k)$.

Alternatively, (and the way you'd prove the statement ...$\Rightarrow a_k < k$) as I intended to write), here's the proof using the hint:

Let $i \in \mathbb{N}$ be arbitrary.
  Then either all elements in the sequence equal a_i or not.
  Case 1: Everything in sequence is the same, i.e., $\forall j \in \mathbb{N}, a_i = a_j$.
    Then let $k = a_i + 1$. So $k \in \mathbb{N}$.
    Also, $a_k = a_i < a_i + 1 = k$.
    Thus $\exists k \in \mathbb{N}, (a_i = a_k \Rightarrow a_i < k)$
  Case 2: Not everything in sequence the same, i.e., $\exists j \in \mathbb{N}, a_i \neq a_j$
    Consider $j \in \mathbb{N}$ such that $a_i \neq a_j$.
    Then $a_i = a_j \Rightarrow a_i < j$ (since $a_i = a_j$ is false).
    So $\exists j \in \mathbb{N}, (a_i = a_j \Rightarrow a_i < j)$.
  So, in each case, $\exists k \in \mathbb{N}, (a_i = a_k \Rightarrow a_i < k)$.
Since i is an arbitrary element of $\mathbb{N}$, $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, (a_i = a_k \Rightarrow a_i < k)$.