University of Toronto, St. George Campus
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Quiz #5 Sample Solutions

We want to prove the following statement:

$$T(n) \; \in \; O(n)$$

where $T(n)$ is the worst case number of lines executed by the algorithm on an array of length n.

We construct the standard proof structure and fill it in, choosing values for c and B when needed:

Let $c = 6$. Then $c \in \mathbb{R}^+$.
  Let $B = 1$. Then $B \in \mathbb{N}$.
    Let $n \in \mathbb{N}$ be arbitrary.
      Assume $n \geq B$.
        From the algorithm, the loop executes no more than $n/2$ times on an array of n integers.
        In a single iteration of the loop, no more than 5 lines are executed (the if statement and the increment of i) in the body, plus the while statement itself.
        Also, 2 lines are executed before the loop, and we need to count the while line when the test fails.
        Thus, $T(n) \leq 2 + (n/2) \times 6 + 1 \leq 3n + 3 \leq 6n$ (since $n \geq 1$).
        [we go back and fill in values for c and B now]
        So $T(n) \leq cn$.
      So $n \geq B \Rightarrow T(n) \leq cn$.
    Since n is an arbitrary element of $\mathbb{N}$, $\forall n \in \mathbb{N}$, $n \geq B \Rightarrow T(n) \leq cn$.
  Since B is a natural number, $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow T(n) \leq cn$.
Since c is a real positive number, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow T(n) \leq cn$.
By definition, $T(n) \in O(n)$.

Note that many other values for c and B also work. We need to be careful that in our count of lines that could be executed, we truly over-estimate (or exactly count) the number of lines that could actually be executed by the algorithm running on a real input.
If we were trying to prove $T(n) \in \Omega(n)$, then we would find an actual input array A on which the algorithm executed more than cn lines (for some c), thus showing $T(n) \geq t(A) \geq cn$. In our proof though, showing how long the algorithm takes on a specific input is not useful (even if we showed it took less than cn lines), since the worst case is bigger than that (and possibly bigger than cn), i.e., the inequality goes the wrong way.