University of Toronto, St. George Campus
Department of Computer Science

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Quiz #3 Sample Solutions

We want to prove the following statement:

$$[(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} \geq 2a_i)] \Rightarrow (\exists i \in \mathbb{N}, a_i \geq 5)$$

The first thing we should do is determine a valid structure for the proof.

Assume $(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} \geq 2a_i)$.
  Then $a_1 \geq 1$ (by $\wedge$E).
  And $\forall i \in \mathbb{N}, a_{2i} \geq 2a_i$ (by $\wedge$E).
  Let $i =$ ____.
    Then $i \in \mathbb{N}$.
    ⋮
    Thus $a_i \geq 5$ (by ____).
  Thus $\exists i \in \mathbb{N}, a_i \geq 5$ (by $\exists$I).
Therefore, $[(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} \geq 2a_i)] \Rightarrow (\exists i \in \mathbb{N}, a_i \geq 5)$ (by $\Rightarrow$I).

We're most of the way there now. We just need to pick a value for i that will let us prove $a_i \geq 5$. Now is the time for a little thought. What value will work? If we're clever, we can probably figure it out.
Or we can see what that universal statement will tell us. We already know something about a_1, so wouldn't it be nice to learn something about a_2? That universal will work nicely!

$1 \in \mathbb{N}$ (by definition of natural numbers).
$a_2 \geq 2a_1$ (by $\forall$E for $i = 1$).
So $a_2 \geq 2 \cdot 1 = 2$ (by transitivity of $\geq$).

We can go on for 2 to learn about a_4, then about a_8:

$2 \in \mathbb{N}$ (by definition of natural numbers).
$a_4 \geq 2a_2$ (by $\forall$E for $i = 2$).
So $a_4 \geq 2 \cdot 2 = 4$ (by transitivity of $\geq$).
$4 \in \mathbb{N}$ (by definition of natural numbers).
$a_8 \geq 2a_4$ (by $\forall$E for $i = 4$).
So $a_8 \geq 2 \cdot 4 = 8$ (by transitivity of $\geq$).

So I guess we should pick $i = 8$ in our proof. Here's the completed version:

Assume $(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} \geq 2a_i)$.
  Then $a_1 \geq 1$ (by $\wedge$E).
  And $\forall i \in \mathbb{N}, a_{2i} \geq 2a_i$ (by $\wedge$E).
  Since $1 \in \mathbb{N}$, $a_2 \geq 2a_1$ (by $\forall$E). So $a_2 \geq 2 \cdot 1 = 2$ (by transitivity of $\geq$).
  Since $2 \in \mathbb{N}$, $a_4 \geq 2a_2$ (by $\forall$E). So $a_4 \geq 2 \cdot 2 = 4$ (by transitivity of $\geq$).
  Since $4 \in \mathbb{N}$, $a_8 \geq 2a_4$ (by $\forall$E). So $a_8 \geq 2 \cdot 4 = 8$ (by transitivity of $\geq$).
  Let $i = 8$.
    Then $i \in \mathbb{N}$.
    Since $8 \geq 5$ (by math), we get $a_i \geq 5$ (by transitivity of $\geq$).
  Thus $\exists i \in \mathbb{N}, a_i \geq 5$ (by $\exists$I).
Therefore, $[(a_1 \geq 1) \wedge (\forall i \in \mathbb{N}, a_{2i} \geq 2a_i)] \Rightarrow (\exists i \in \mathbb{N}, a_i \geq 5)$ (by $\Rightarrow$I).

Note that it's okay to put our reasoning about $a_2, \ldots, a_8$ in the ⋮ part of our original proof outline instead.