PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2006 EXAMINATIONS

CSC 165 H1F
Instructor: R. Krueger

Duration — 3 hours

No Aids Allowed

PLEASE HAND IN

Student Number: |_|_|_|_|_|_|_|_|_|

Last (Family) Name(s): SOLUTIONS

First (Given) Name(s): SAMPLE

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above,
and read the instructions below *carefully*.)

This final examination consists of 7 questions on 14 pages (including this one), printed on one side of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete and write your student number where indicated at the bottom of every page* (except page 1).

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

When giving a symbolic statement, if there is more than one interpretation, be sure to use parentheses to make your answer clear.

Where a proof is required, if your proof is well-structured and clear then the standard ending summaries for quantification and implication are not required. You will earn substantial part marks for writing the outline of the proof and indicating which steps are missing.

Marking Guide

1: ______/10

2: ______/12

3: ______/12

4: ______/12

5: ______/25

6: ______/ 6

7: ______/18

TOTAL: ______/95

Good Luck!

Question 1. Orchestrating success. [10 MARKS]

Let domain S be the set of students,
domain O be the set of orchestras,
domain E be the set of the exams,
predicate $cello(x)$ represent "student x plays the cello,"
predicate $study(x, y)$ represent "student x studies for exam y," and
predicate $orchestra(x, y)$ represent "student x plays in orchestra y."

Write each of the following statements symbolically. Do not introduce any new domains or predicates.

Every student studies for the CSC 165 exam.

SAMPLE SOLUTION

$\forall s \in S, study(s, \text{CSC165})$

[1 mark]

Only one student in the class plays the cello.

SAMPLE SOLUTION

$\exists s \in S, cello(s) \wedge \forall x \in S, cello(x) \Rightarrow x = s$

[2 marks: 1 for each half ("at least one" and "at most one")]

Claire (a student) can play in the TSO (Toronto Symphony Orchestra) only if she plays the cello.

SAMPLE SOLUTION

$orchestra(\text{Claire}, \text{TSO}) \Rightarrow cello(\text{Claire})$

[2 marks: 1 for lack of quantification, 1 for direction of implication]

Most, but not all, students study for the CSC 165 exam.

SAMPLE SOLUTION

$\exists x \in S, study(x, \text{CSC165}) \wedge \exists y \in S, \neg study(y, \text{CSC165})$

[2 marks: 1 for each half ("there are some who do" and "there are some who don't")]
Note: our logic doesn't have a way to express the imprecise quantity "most" — does it mean more than half? three-quarters? all but a few? all but one?

Only students that don't study for any exams can play in an orchestra.

SAMPLE SOLUTION

$\forall s \in S, (\exists o \in O, orchestra(s, o)) \Rightarrow (\forall e \in E, \neg study(s, e))$

[3 marks: 1 each half, 1 for direction of implication]

Question 2. Iff I'm right. [12 MARKS]

Consider:

$$\text{(S1)} \quad (A \Rightarrow B) \Leftrightarrow C$$
$$\text{(S2)} \quad A \Rightarrow (B \Leftrightarrow C)$$

Part (a) [4 MARKS]

Give truth values for A, B and C such that one of (S1), (S2) is true and the other is false. Briefly justify.

SAMPLE SOLUTION

The truth table for (S1) and (S2) is as follows:

A	B	C	(S1)	(S2)
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	F	T

Thus, if we set A and C to false and set B to true, (S1) would be false (false implies true is true, but C is false) while (S2) would be true (A is false, so the implication is vacuously true).

MARKING SCHEME:

- 1 mark: using a reasonable strategy to answer question, or coming up with correct answer
- 2 marks: giving correct truth values for A,B,C
- 1 mark: convincing/coherent justification that statements have different truth values

Part (b) [2 MARKS]

Suppose we know that A is true. Show that (S1) and (S2) are now equivalent. (A formal proof is not necessary.)

SAMPLE SOLUTION

We only look at the first four rows of the truth table (when A is true). (S1) and (S2) have the same truth value.

Alternatively, substitute "true" in the formulae where A appears, and simplify. We are left with $B \Leftrightarrow C$ for each formula.

Part (c) [3 MARKS]
Write the negation of (S1) symbolically. Move any negations inside as far as possible.

SAMPLE SOLUTION

First step is $((A \Rightarrow B) \wedge \neg C) \vee (\neg(A \Rightarrow B) \wedge C)$.
Moving negations inside, answer is $((A \Rightarrow B) \wedge \neg C) \vee (A \wedge \neg B \wedge C)$

MARKING SCHEME:

- 1 mark: understanding how to negate $\Leftrightarrow$
- 2 marks: correct manipulation moving negations in

Part (d) [3 MARKS]
Write the contrapositive of (S2) symbolically. Move any negations inside as far as possible.

SAMPLE SOLUTION

First step is $\neg(B \Leftrightarrow C) \Rightarrow \neg A$
Moving negations inside, answer is $(B \wedge \neg C \vee C \wedge \neg B) \Rightarrow \neg A$

MARKING SCHEME:

- 1 mark: first step correct (understanding contrapositive)
- 2 marks: correct manipulation moving negations in

Question 3. Next in line, please. [12 MARKS]

Consider an infinite sequence of real numbers $a_0, a_1, \ldots$.

In your answers below, use only the (in)equality predicates $<, \leq, =, \neq, \geq, >$ and the domains $\mathbb{R}$, $\mathbb{N}$. Do not use any other predicates, domains, or any functions (like min or max).

Part (a) [2 MARKS]

Express symbolically: "u is an upper bound for the sequence"
(Recall that u is an upper bound if nothing is greater than u.)

SAMPLE SOLUTION

$\forall i \in \mathbb{N}, a_i \leq u$

Part (b) [3 MARKS]

Express symbolically: "m is the maximum value in the sequence"

SAMPLE SOLUTION

$(\exists j \in \mathbb{N}, m = a_j) \wedge (\forall i \in \mathbb{N}, a_i \leq m)$

or

$\exists j \in \mathbb{N}, m = a_j \wedge \forall i \in \mathbb{N}, a_i \leq a_j$

Part (c) [3 MARKS]

Express symbolically: "there is no upper bound for the sequence"

SAMPLE SOLUTION

$\forall u \in \mathbb{R}, \exists i \in \mathbb{N}, a_i > u$

or

$\neg \exists u \in \mathbb{R}, \forall i \in \mathbb{N}, a_i \leq u$

Part (d) [4 MARKS]

Outline the proof structure you would use to prove the following statement. Do not actually prove it!

$$(\exists x \in \mathbb{R}, \forall i \in \mathbb{N}, a_i \leq x) \Rightarrow (\forall j \in \mathbb{N}, \exists y \in \mathbb{N}, (a_j)^j \leq y)$$

SAMPLE SOLUTION

Assume $\exists x \in \mathbb{R}, \forall i \in \mathbb{N}, a_i \leq x$.
    Let $j \in \mathbb{N}$ be arbitrary.
        Let $y =$ ____. Then $y \in \mathbb{N}$.
            ⋮
            So $(a_j)^j \leq y$.
        Since y is a particular element in $\mathbb{N}$, $\exists y \in \mathbb{N}, (a_j)^j \leq y$.
    Since j is an arbitrary element of $\mathbb{R}$, $\forall j \in \mathbb{N}, \exists y \in \mathbb{N}, (a_j)^j \leq y$.
Thus $(\exists x \in \mathbb{R}, \forall i \in \mathbb{N}, a_i \leq x) \Rightarrow (\forall j \in \mathbb{N}, \exists y \in \mathbb{N}, (a_j)^j \leq y)$.

MARKING SCHEME:

- 1 mark: Assuming hypothesis
- 1 mark: noting $y \in \mathbb{N}$
- 2 marks: overall correctness

Question 4. Biochemical dreams. [12 MARKS]

Biochemistry researchers often want to create random DNA nucleotide sequences. These sequences are comprised of four nucleotides, A = adenine, C = cytosine, G = guanine, and T = thymine. In order for these random chains to be useful, they must obey the property that no two consecutive nucleotides can be the same.

Suppose we have a subsequence of nucleotides x, y, z where $x, y, z \in \{A, C, G, T\}$.

Let $P(x, y, z)$ mean "The subsequence of nucleotides x, y, z is a valid subsequence."

Part (a) [6 MARKS]

Prove the following statement using our structured form:

$\forall x \in \{A, C, G, T\}, \forall z \in \{A, C, G, T\}, \exists y \in \{A, C, G, T\},\ P(x, y, z)$

SAMPLE SOLUTION

Let $x \in \{A, C, G, T\}$.
  Let $z \in \{A, C, G, T\}$.
    We now need to show an acceptable y exists. We can show this using cases.
    We start with $x = A \vee x \neq A$.
    Case $x = A$:
      Then $z = C \vee z \neq C$.
      Case $z = C$:
        Let $y = G$. Then $y \in \{A, C, G, T\}$.
        Since $x \neq y$ and $z \neq y$, $P(x, y, z)$.
        Thus $\exists y \in \{A, C, G, T\}, P(x, y, z)$.
      Case $z \neq C$:
        Let $y = C$. Then $y \in \{A, C, G, T\}$.
        Since $x \neq y$ and $z \neq y$, $P(x, y, z)$.
        Thus $\exists y \in \{A, C, G, T\}, P(x, y, z)$.
    Case $x \neq A$:
      Then $z = A \vee z \neq A$.
      Case $z \neq A$:
        Let $y = A$. Then $y \in \{A, C, G, T\}$.
        And $x \neq y$ and $z \neq y$, so $P(x, y, z)$.
        Thus $\exists y \in \{A, C, G, T\}, P(x, y, z)$.
      Case $z = A$:
        Then $x = C \vee x \neq C$.
        Case $x = C$:
          Let $y = G$. Then $y \in \{A, C, G, T\}$, and since $x \neq y$ and $z \neq y$, $P(x, y, z)$.
          Thus $\exists y \in \{A, C, G, T\}, P(x, y, z)$.
        Case $x \neq C$:
          Let $y = C$. Then $y \in \{A, C, G, T\}$, and since $y \neq x$ and $y \neq z$, $P(x, y, z)$.
          Thus $\exists y \in \{A, C, G, T\}, P(x, y, z)$.
    In all cases we have shown $\exists y \in \{A, C, G, T\}, P(x, y, z)$.
Thus $\forall x \in \{A, C, G, T\}, \forall z \in \{A, C, G, T\}, \exists y \in \{A, C, G, T\},\ P(x, y, z)$.

Alternatively, you can argue that after disallowing 2 choices for y, there is at least one acceptable choice remaining. Technically, this method reduces to precisely the above case analysis.

MARKING SCHEME:

- 3 marks: structure
- 3 marks: argument
- omitting some of the repetition (as long as overall structure is correct and omitted cases are similarly provable) is acceptable

Part (b) [6 MARKS]

Disprove the following statement using our structured form:

$\forall x \in \{A, C, G, T\}, \exists y \in \{A, C, G, T\}, \forall z \in \{A, C, G, T\},\ P(x, y, z)$

SAMPLE SOLUTION

We need to show that $\exists x \in \{A, C, G, T\}, \forall y \in \{A, C, G, T\}, \exists z \in \{A, C, G, T\},\ \neg P(x, y, z)$

Let $x = A$. Then $x \in \{A, C, G, T\}$.

Let $y \in \{A, C, G, T\}$ be arbitrary.

Let $z = y$.

Then $z \in \{A, C, G, T\}$.

Since $z = y$, it must be that $\neg P(x, y, z)$.

Hence $\exists z \in \{A, C, G, T\},\ \neg P(x, y, z)$.

Hence $\forall y \in \{A, C, G, T\}, \exists z \in \{A, C, G, T\},\ \neg P(x, y, z)$.

Hence $\exists x \in \{A, C, G, T\}, \forall y \in \{A, C, G, T\}, \exists z \in \{A, C, G, T\},\ \neg P(x, y, z)$.

MARKING SCHEME:

- 3 marks: structure
- 3 marks: argument

Question 5. Big Ohs. [25 MARKS]

Remember that you may omit the ending summaries if your proof is well-structured and clear.

Part (a) [8 MARKS]

Prove or disprove: $\frac{2n^3+165}{n+1} \notin O(n)$.

SAMPLE SOLUTION

We prove the statement $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge \frac{2n^3+165}{n+1} > cn$.

Let $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$ be arbitrary.

Let $n = \max\{\lceil c \rceil + 1, B\}$. Then $n \in \mathbb{N}$.

Thus $n \geq B$.

Also, $n > c$ and $n > 1$.

So $\frac{2n^3+165}{n+1} > \frac{2n^3}{n+1} > \frac{2n^3}{n+n} = \frac{2n^3}{2n} = n^2$.

Since $n^2 > cn$, we conclude $\frac{2n^3+165}{n+1} > cn$.

Therefore $\exists n \in \mathbb{N}, n \geq B \wedge \frac{2n^3+165}{n+1} > cn$.

Therefore $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge \frac{2n^3+165}{n+1} > cn$.

Thus $\frac{2n^3+165}{n+1} \notin O(n)$.

MARKING SCHEME:

- 2 marks: proving $\notin$ and statement of this
- 2 marks: overall structure
- 1 mark: choice of n, showing $n \in \mathbb{N}$
- 3 marks: argument

Part (b) [8 MARKS]

Stirling's approximation for the factorial function says that $\log_2(n!) \approx n \log_2 n - n$.
Use this fact to prove that $n! \in \Omega(2^n)$. *Hint:* If $x \geq 1$, then $x^y \geq x$ when $y \geq 1$.

SAMPLE SOLUTION

Let $c = 1$. Then $c \in \mathbb{R}^+$.
Let $B = 8$. Then $B \in \mathbb{N}$.
Let $n \in \mathbb{N}$ be arbitrary.
Assume $n \geq B$.
Then $n \geq 8$.
Then $\log_2 n \geq \log_2 8 = 3$.
Then $n \log_2 n \geq 3n$ since $n \geq 1$.
Then $n \log_2 n - n \geq 2n$ (subtracting n from each side).
Since $\log_2(n!) \approx n \log_2 n - n$, surely $\log_2(n!) \geq n$ (the approximation is greater than $2n$, so surely greater than n, since $n \geq 8$).
So $n! = 2^{(\log_2(n!))} \geq 2^n$.
Thus $n! \geq 2^n = c2^n$.
Thus $n \geq B \Rightarrow n! \geq c2^n$.
Therefore $\forall n \in \mathbb{N}, n \geq B \Rightarrow n! \geq c2^n$.
Therefore $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow n! \geq c2^n$.
Therefore $\forall c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow n! \geq c2^n$.
Thus $n! \in \Omega(2^n)$.

MARKING SCHEME:

- 2 marks: overall structure
- 2 marks: choice of c, B, showing domains
- 4 marks: argument

Part (c) [9 MARKS]
Prove that $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, (a > 0 \wedge b > 0) \Rightarrow (n+a)^b \in \Theta(n^b)$.

SAMPLE SOLUTION

Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ be arbitrary.
    Assume $a > 0 \wedge b > 0$.
        Let $c_1 = 1$. Then $c_1 \in \mathbb{R}^+$.
            Let $c_2 = 2^b$. Then $c_2 \in \mathbb{R}^+$.
                Let $B = \lceil a \rceil$. Since $a > 0$, $B \in \mathbb{N}$.
                    Let $n \in \mathbb{N}$ and assume $n \geq B$.
                        Then $c_1 n^b = n^b \leq (n+a)^b$ since $a > 0$ and $b > 0$.
                        Also, $(n+a)^b \leq (n+n)^b = (2n)^b \leq 2^b n^b = c_2 n^b$.
                        Thus $c_1 n^b \leq (n+a)^b \leq c_2 n^b$.
                ...
        Thus $(n+a)^b \in \Theta(n^b)$.
Thus $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, (a > 0 \wedge b > 0) \Rightarrow (n+a)^b \in \Theta(n^b)$.

MARKING SCHEME:

- 1 mark for "Let a, b"
- 1 mark for "assume $a > 0 \wedge b > 0$"
- 2 marks overall Θ proof structure
- 1 mark for B selection, showing in $\mathbb{N}$
- 2 marks for c_1 argument
- 2 marks for c_2 argument

Question 6. Follow the recipe. [6 MARKS]

Consider the following method (assume $n > 0$ and $k > 0$):

```
int divide(int n, int k) {  // Precondition: n > 0 and k > 0
  int x = n;
  int r = 0;
  while ( x >= k ) {  // Invariant: r = (n - x) / k
    r++;
    x = x - k;
  }
  return r;
}
```

Suppose someone tells you that the loop invariant is $r = (n-x)/k$. Prove that, if the claimed loop invariant is true at the *beginning* of one loop iteration, then it is true at the *end* of that one loop iteration.

SAMPLE SOLUTION

Let $n \in \mathbb{N}$ and let $k \in \mathbb{N}$. Assume $n > 0 \wedge k > 0$.

Let $r \in \mathbb{Z}$ and $x \in \mathbb{Z}$. [they'll be the values of the variables at the start of an execution of the loop]

Let r' and x' represent the values of the variables at the end of the loop, related to r and x by the execution of the program.

Assume $r = (n-x)/k$.

Assume $x \geq k$. Then the loop body will be executed.

So $x' = x - k$, and so $x = x' + k$.

So $r' = r + 1$ by action of algorithm

$= \frac{n-x}{k} + 1$ by invariant assumption

$= \frac{n-(x'+k)}{k} + 1$ since $x = x' + k$

$= \frac{n-x'}{k} - 1 + 1$

$= \frac{n-x'}{k}$

Thus $r = (n-x)/k \wedge x \geq k$ at start of an iteration $\Rightarrow r = (n-x)/k$ at the end of the iteration

Thus, $\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, n > 0 \wedge k > 0 \Rightarrow$ for all $r \in \mathbb{Z}$ and $x \in \mathbb{Z}$ computed by the algorithm, $r = (n-x)/k \wedge x \geq k$ at start of an iteration $\Rightarrow r = (n-x)/k$ at the end of the iteration

MARKING SCHEME:

- 2 marks: reasonable definitions
- 1 mark: showing implication to prove (explicit or implicit), structure
- 3 marks: argument

Question 7. Floating point systems. [18 MARKS]

Part (a) [14 MARKS]

Consider the normalized floating point system F with $\beta = 2$, $t = 3$, $e_{\text{max}} = 2$ and $e_{\text{min}} = -2$.

(i) How many positive (and non-zero) numbers are there in F? (You do not need to simplify your answer.)

[1 mark] Positive numbers are of the form $+1.xx \times 2^e$, where $e \in \{2, 1, 0, -1, -2\}$.
There are $2^2 \times 5 = 20$ positive numbers in F.

(ii) Give the smallest and largest positive (and non-zero) numbers representable in F.

SAMPLE SOLUTION

[2 marks]
Smallest is $(1.00)_2 \times 2^{-2} = 1/4$ and largest is $(1.11)_2 \times 2^2 = 7$

(iii) Give all the real numbers that can be represented exactly in F within the real interval $[\frac{3}{4}, 1]$. (You may leave your answers as fractions.)

SAMPLE SOLUTION

[3 marks]
$3/4 = (1.10)_2 \times 2^{-1}$
$7/8 = (1.11)_2 \times 2^{-1}$
$1 = (1.00)_2 \times 2^0$

(iv) Represent 1.35 and 2.2 (both currently written in base 10) in the system F by rounding to the *nearest* number in the system.

SAMPLE SOLUTION

[4 marks]
$(1.01)_2 \times 2^0 = 1.25$ and $(1.10)_2 \times 2^0 = 1.5$.
Hence 1.35 would be represented by $(1.01)_2 \times 2^0$.

$(1.00)_2 \times 2^1 = 2.0$ and $(1.01)_2 \times 2^1 = 2.5$.
Hence 2.2 would be represented by $(1.00)_2 \times 2^1$.

(v) Give the relative error for each representation in (iv), expressed as a fraction in lowest terms.

Sample Solution

[4 marks]
$re_{1.35} = \frac{|1.35-1.25|}{|1.35|} = \frac{10}{135} = \frac{2}{27}$
$re_{2.2} = \frac{|2.2-2.0|}{|2.2|} = \frac{2}{22} = \frac{1}{11}$

Part (b) [4 marks]

Suppose we need to represent 3.125 and 3.1875 *exactly* using base $\beta = 2$. Find the smallest number of digits t and range of exponents $e_{\max}, \ldots, e_{\min}$ required. Give the representation of these numbers in your system. (*Hint:* $2^{-3} = 0.125$ and $2^{-4} = 0.0625$.)

Sample Solution

$3.125 = (11.001)_2 = (1.1001)_2 \times 2^1$ and $3.1875 = (11.0011)_2 = (1.10011)_2 \times 2^1$.

Therefore, we need $t = 6$ digits to represent these numbers exactly.
We could set $e_{\max} = e_{\min} = 1$.