PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2006 EXAMINATIONS

CSC 165 H1F
Instructor: R. Krueger

Duration — 3 hours

No Aids Allowed

PLEASE HAND IN

Student Number: _ _ _ _ _ _ _ _ _

Last (Family) Name(s): ____________________

First (Given) Name(s): ____________________

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below *carefully*.)

This final examination consists of 7 questions on 11 pages (including this one), printed on one side of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete and write your student number where indicated at the bottom of every page* (except page 1).

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

When giving a symbolic statement, if there is more than one interpretation, be sure to use parentheses to make your answer clear.

Where a proof is required, if your proof is well-structured and clear then the standard ending summaries for quantification and implication are not required. You will earn substantial part marks for writing the outline of the proof and indicating which steps are missing.

Marking Guide

# 1:	____/10
# 2:	____/12
# 3:	____/12
# 4:	____/12
# 5:	____/25
# 6:	____/ 6
# 7:	____/18
TOTAL:	____/95

Good Luck!

Question 1. Orchestrating success. [10 MARKS]

Let domain S be the set of students,
domain O be the set of orchestras,
domain E be the set of the exams,
predicate $cello(x)$ represent "student x plays the cello,"
predicate $study(x, y)$ represent "student x studies for exam y," and
predicate $orchestra(x, y)$ represent "student x plays in orchestra y."

Write each of the following statements symbolically. Do not introduce any new domains or predicates.

Every student studies for the CSC 165 exam.

Only one student in the class plays the cello.

Claire (a student) can play in the TSO (Toronto Symphony Orchestra) only if she plays the cello.

Most, but not all, students study for the CSC 165 exam.

Only students that don't study for any exams can play in an orchestra.

Question 2. Iff I'm right. [12 MARKS]

Consider:

$$\text{(S1)} \quad (A \Rightarrow B) \Leftrightarrow C$$
$$\text{(S2)} \quad A \Rightarrow (B \Leftrightarrow C)$$

Part (a) [4 MARKS]

Give truth values for A, B and C such that one of (S1), (S2) is true and the other is false. Briefly justify.

Part (b) [2 MARKS]

Suppose we know that A is true. Show that (S1) and (S2) are now equivalent. (A formal proof is not necessary.)

Part (c) [3 MARKS]

Write the negation of (S1) symbolically. Move any negations inside as far as possible.

Part (d) [3 MARKS]

Write the contrapositive of (S2) symbolically. Move any negations inside as far as possible.

Question 3. Next in line, please. [12 MARKS]

Consider an infinite sequence of real numbers $a_0, a_1, \ldots$.

In your answers below, use only the (in)equality predicates $<, \leq, =, \neq, \geq, >$ and the domains $\mathbb{R}$, $\mathbb{N}$. Do not use any other predicates, domains, or any functions (like min or max).

Part (a) [2 MARKS]

Express symbolically: "u is an upper bound for the sequence"
(Recall that u is an upper bound if nothing is greater than u.)

Part (b) [3 MARKS]

Express symbolically: "m is the maximum value in the sequence"

Part (c) [3 MARKS]

Express symbolically: "there is no upper bound for the sequence"

Part (d) [4 MARKS]

Outline the proof structure you would use to prove the following statement. Do not actually prove it!

$$(\exists x \in \mathbb{R}, \forall i \in \mathbb{N}, a_i \leq x) \Rightarrow (\forall j \in \mathbb{N}, \exists y \in \mathbb{N}, (a_j)^j \leq y)$$

Question 4. Biochemical dreams. [12 MARKS]

Biochemistry researchers often want to create random DNA nucleotide sequences. These sequences are comprised of four nucleotides, A = adenine, C = cytosine, G = guanine, and T = thymine. In order for these random chains to be useful, they must obey the property that no two consecutive nucleotides can be the same.

Suppose we have a subsequence of nucleotides x, y, z where $x, y, z \in \{A, C, G, T\}$.

Let $P(x, y, z)$ mean "The subsequence of nucleotides x, y, z is a valid subsequence."

Part (a) [6 MARKS]

Prove the following statement using our structured form:

$\forall x \in \{A, C, G, T\}, \forall z \in \{A, C, G, T\}, \exists y \in \{A, C, G, T\},\ P(x, y, z)$

Part (b) [6 MARKS]

Disprove the following statement using our structured form:

$\forall x \in \{A, C, G, T\}, \exists y \in \{A, C, G, T\}, \forall z \in \{A, C, G, T\},\ P(x, y, z)$

Question 5. Big Ohs. [25 MARKS]

Remember that you may omit the ending summaries if your proof is well-structured and clear.

Part (a) [8 MARKS]

Prove or disprove: $\frac{2n^3+165}{n+1} \notin O(n)$.

Part (b) [8 MARKS]

Stirling's approximation for the factorial function says that $\log_2(n!) \approx n \log_2 n - n$.
Use this fact to prove that $n! \in \Omega(2^n)$. *Hint:* If $x \geq 1$, then $x^y \geq x$ when $y \geq 1$.

Part (c) [9 MARKS]
Prove that $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, (a > 0 \wedge b > 0) \Rightarrow (n+a)^b \in \Theta(n^b)$.

Question 6. Follow the recipe. [6 MARKS]

Consider the following method (assume $n > 0$ and $k > 0$):

```
int divide(int n, int k) {  // Precondition: n > 0 and k > 0
  int x = n;
  int r = 0;
  while ( x >= k ) {  // Invariant: r = (n - x) / k
    r++;
    x = x - k;
  }
  return r;
}
```

Suppose someone tells you that the loop invariant is $r = (n - x)/k$. Prove that, if the claimed loop invariant is true at the *beginning* of one loop iteration, then it is true at the *end* of that one loop iteration.

Question 7. Floating point systems. [18 MARKS]

Part (a) [14 MARKS]

Consider the normalized floating point system F with $\beta = 2$, $t = 3$, $e_{\max} = 2$ and $e_{\min} = -2$.

(i) How many positive (and non-zero) numbers are there in F? (You do not need to simplify your answer.)

(ii) Give the smallest and largest positive (and non-zero) numbers representable in F.

(iii) Give all the real numbers that can be represented exactly in F within the real interval $[\frac{3}{4}, 1]$. (You may leave your answers as fractions.)

(iv) Represent 1.35 and 2.2 (both currently written in base 10) in the system F by rounding to the *nearest* number in the system.

(v) Give the relative error for each representation in (iv), expressed as a fraction in lowest terms.

Part (b) [4 MARKS]

Suppose we need to represent 3.125 and 3.1875 *exactly* using base $\beta = 2$. Find the smallest number of digits t and range of exponents $e_{\text{max}}, \ldots, e_{\text{min}}$ required. Give the representation of these numbers in your system. (*Hint:* $2^{-3} = 0.125$ and $2^{-4} = 0.0625$.)

Total Marks = 95