Note to Students: This file contains sample solutions to the term test together with the marking scheme and comments for each question. Please read the solutions and the marking schemes and comments carefully. Make sure that you understand why the solutions given here are correct, that you understand the mistakes that you made (if any), and that you understand *why* your mistakes were mistakes.

Remember that although you may not agree completely with the marking scheme given here it was followed the same way for all students. We will remark your test only if you clearly demonstrate that the marking scheme was not followed correctly. We will make *no* exception to the marking scheme, unless you can clearly demonstrate that it is somehow incorrect.

For all remarking requests, please submit your request **in writing** directly to your instructor. For all other questions, please don't hesitate to ask your instructor during office hours or by e-mail.

Question 1. [6 MARKS]

Consider the following statement:

(S) If A, then B and C.

Part (a) [1 MARK]

What, if anything, can be concluded from (S) if A is true?

SAMPLE SOLUTION

Following the implication in (S), we know both B and C are true.

Part (b) [1 MARK]

What, if anything, can be concluded from (S) if B is true and C is true?

SAMPLE SOLUTION

Nothing! The implication says nothing about A (B and C could be true without A necessarily being true).

Part (c) [2 MARKS]

Express symbolically a statement equivalent to (S), but without using implication ($\Rightarrow$).

SAMPLE SOLUTION

We use the implication rule to get an equivalent formula using or:
$\neg A \vee (B \wedge C)$

Part (d) [2 MARKS]

Express symbolically a statement equivalent to (S), but without using conjunction ($\wedge$).

SAMPLE SOLUTION

We use DeMorgan's laws to replace the and with an or:
$A \implies \neg(\neg B \vee \neg C)$

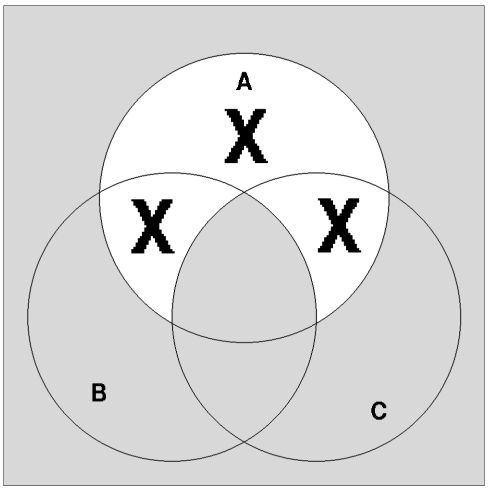
Question 2. [4 MARKS]

For each of the following statements, draw a Venn diagram with overlapping circles for A, B and C, making 8 regions. Shade in all the regions where (S) is true, and put an X in each region where (S) is false.

If A, then B and C.

SAMPLE SOLUTION

Symbolically, this means $A \Rightarrow (B \wedge C)$.



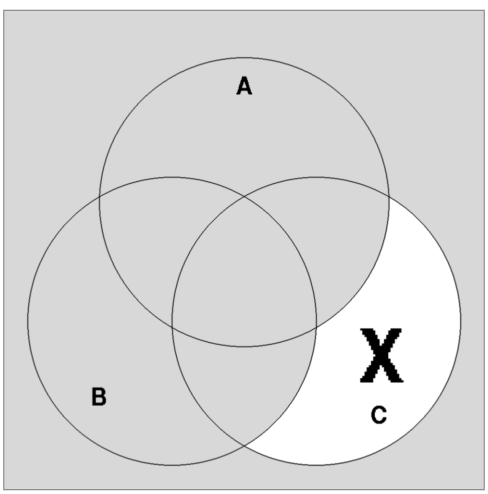
MARKING SCHEME:

A. 2 marks: correctly and fully filling diagram
- -0.5 for forgetting outer region

A or B are true, when C is true.

SAMPLE SOLUTION

Symbolically, this means $C \Rightarrow (A \vee B)$.



MARKING SCHEME:

A. 2 marks: correctly and fully filling diagram
- 1/2 for correctly drawing $(A \vee B) \implies C$

Question 3. [8 MARKS]

Let F be a set of friends, and let $P(x, y)$ mean "x pays for y."
Express each of the following statements symbolically, matching the English form as closely as possible.

Part (a) [2 MARKS]

Everybody pays for at least one person.

SAMPLE SOLUTION

$\forall a \in F, \exists b \in F, P(a, b)$

Part (b) [2 MARKS]

Everyone has somebody who pays for him.

SAMPLE SOLUTION

$\forall a \in F, \exists b \in F, P(b, a)$

MARKING SCHEME:

A. 2 marks: fundamental correctness
- 0/2 if quantifiers swapped
- 1/2 if $P(b, a)$ accidentally written as $P(a, b)$

Part (c) [2 MARKS]

Somebody pays for him or herself.

SAMPLE SOLUTION

$\exists a \in F, P(a, a)$

Part (d) [2 MARKS]

Nobody pays for somebody else.

SAMPLE SOLUTION

$\neg \exists a \in F, \exists b \in F, P(a, b) \wedge a \neq b$
or
$\forall a \in F, \forall b \in F, P(a, b) \implies a = b$ or $\forall a \in F, \forall b \in F, a \neq b \implies \neg P(a, b)$

MARKING SCHEME:

A. 2 marks: fundamental correctness
- 1/2 for missing the "else"-ness ($a = b$ or $a \neq b$)

Question 4. [8 MARKS]

Consider the following statement about sequences of natural numbers $a_0, a_1, a_2, \ldots$ (recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$):

(T) $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, (a_i = a_k \implies a_i < k)$

Part (a) [2 MARKS]

Write the negation of (T) symbolically, moving negations inside as much as possible.

SAMPLE SOLUTION

$\exists i \in \mathbb{N}, \forall k \in \mathbb{N}, a_i = a_k \wedge a_i \geq k$

MARKING SCHEME:

A. 1 mark: negating quantifiers
B. 1 mark: negating implication
- writing $\neg(a_i < k)$ instead of $a_i \geq k$ is fine

Part (b) [4 MARKS]

For the following sequences, state whether (T) is true or false. Justify your claim, using an example or counterexample when appropriate.

$1, 2, 3, 4, 5, 6, 7, \ldots$

SAMPLE SOLUTION

True. Given any element $i \in \mathbb{N}$, then actually any number $k \neq i$ satisfies the rest of (T): the first part of the implication is false (no repeated elements, so the equality is false), thus the implication is true. Since it holds for at least one number, (T) is true.

$1, 2, 1, 1, 2, 1, 1, 1, 2, \ldots$

SAMPLE SOLUTION

True. Pick any element $i \in \mathbb{N}$. We can pick any k such that $a_k \neq a_i$. Then the first part of the implication is false, so the implication is true.

MARKING SCHEME:

A. 0.5 marks: correctly guessing true/false
B. 1.5 marks: correctly justifying choice with proper structure

Part (c) [2 MARKS]

Give the structure a direct proof of (T) would look like. (Do not try to actually prove (T).)

SAMPLE SOLUTION

Let i be an arbitrary element of $\mathbb{N}$.
  Let $k =$ ____. Then $k \in \mathbb{N}$.
    ⋮
    Then $a_i = a_k \implies a_i < k$.
  Since $k \in \mathbb{N}$, then $\exists k \in \mathbb{N}, (a_i = a_k \implies a_i < k)$.
Since i is an arbitrary element of $\mathbb{N}$, $\forall i \in \mathbb{N}, \exists k \in \mathbb{N}, (a_i = a_k \implies a_i < k)$.

MARKING SCHEME:

A. 2 marks: picking i arbitrary, picking a specific j, and eventually showing the implication