

Due: By 12:00 noon on Thursday, November 2.

Worth: 7%

You must complete and sign an assignment cover page, and attach it (with a staple) to the front of your assignment. Assignments should be handed into the drop box in BA 2220.

1. Consider the sequence of integers defined by:

$$(A) \ a_n = \begin{cases} 1 & \text{if } n \text{ is a multiple of 3 or 4;} \\ 0 & \text{if } n \text{ is not a multiple of 3 or 4.} \end{cases}$$

- (a) Write the first 11 values of the sequence (A) (i.e., $a_0, a_1, \dots, a_{10}$).
 (b) Prove that $\forall i \in \mathbb{N}, a_i \leq a_{i+1} \leq a_{i+2} \vee a_i \geq a_{i+1} \geq a_{i+2}$ is false for (A).
 (c) Prove that $\exists i \in \mathbb{N}, a_i + a_{i+2} + a_{i+4} = 0$ is false for (A).
 (d) Prove that $\forall i \in \mathbb{N}, a_i + a_{i+1} + a_{i+2} < 3$ is true for (A).
2. For $a, b \in \mathbb{N}$, let $P(a, b)$ mean $\exists n \in \{0, 1, 2\}, a = b + n$.

Using our structured form, prove or disprove the following statements:

- (a) $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, \forall k \in \mathbb{N}, P(i, j) \wedge P(i, k) \Rightarrow P(j, k)$
 (b) $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, \exists k \in \mathbb{N}, P(i, j) \vee P(j, i) \Rightarrow P(i, k) \wedge P(j, k)$
 (c) $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, \forall k \in \mathbb{N}, P(i, k) \wedge P(j, k) \Rightarrow P(i, j) \vee P(j, i)$
3. Consider the game of tic-tac-toe. Each player takes turns marking an **X** or **O** on the board in an unoccupied space. If a player has 3 **X**'s or **O**'s in a row, either vertically, horizontally, or diagonally, then that player wins the game.

Consider the following incomplete game, where it's **X**'s turn to move. To help us formalize this game, we number the positions on the board, as illustrated. Let $M = \{1, \dots, 9\}$ be the set of spaces on the board.

1 X	2 O	3 X
4	5	6 O
7	8	9

- (a) Describe, in English, a strategy for **X** that guarantees a win in this game.
 (b) Let $WX(m_1, m_2) = \text{"X wins the game by making moves } m_1 \text{ and } m_2\text{"}$. Write a symbolic expression that describes the values for m_1 and m_2 that make $WX(m_1, m_2)$ true.
 (c) Express symbolically the statement "**X** can win this game."
Hint: Use only the predicate you defined in (b), and make sure that each move is permitted.
 (d) Using our structured form and symbolic language, prove that this game should be a win for **X**. In other words, prove your answer to (c).

4. Consider the following Java pseudo-code that implements a class representing a bag filled with white and black coloured beans.

```
class BeanBag
{
    public static final int WHITE = 0;
    public static final int BLACK = 1;
    public int whitebeans; // white bean counter
    public int blackbeans; // black bean counter

    // removeRandomBean selects a random bean from the bag.
    // If only one coloured bean remains, it returns this colour.
    // the function decrements whitebeans or blackbeans variable.
    public int removeRandomBean();

    // replacebean returns a bean of given colour to the bag.
    // the function increments whitebeans or blackbeans variable.
    public void replaceBean(int beancolour);
}
```

Now consider the following while loop:

```
while (bagofbeans.whitebeans + bagofbeans.blackbeans > 1) do
    int beanonecolour = bagofbeans.removeRandomBean();
    int beantwocolour = bagofbeans.removeRandomBean();
    if (beanonecolour == beantwocolour) then
        bagofbeans.replaceBean(BeanBag.BLACK);
    else
        bagofbeans.replaceBean(BeanBag.WHITE);
    end if
end while
```

- (a) Suppose that there are an odd number of white beans at the beginning of an iteration of the while loop. What can we say about the number of white beans at the end of this iteration of the while loop?
- (b) Using our structured form, prove your answer from part (a).
- (c) Suppose we know that the number of beans eventually decreases to one bean. If there was initially an odd number of white beans, what colour will the remaining bean be? Justify your answer.
- (d) Suppose we know that the number of beans eventually decreases to one bean. If there was initially an even number of white beans, what colour will the remaining bean be? Justify your answer.