1. [13 marks]

   (a) [2 marks] We can state "n is either even or odd" symbolically as
   $(\exists k \in \mathbb{Z}, n = 2k) \vee (\exists k \in \mathbb{Z}, n = 2k + 1)$
   or
   $\exists k \in \mathbb{Z}, (n = 2k \vee n = 2k + 1)$.
   Notice that n is open in this sentence.

   (b) [5 marks] We want to prove that $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 - n = 2k$. We'll prove by cases.

   Assume $n \in Z$.
   Then n is either even or odd.
   That is, $(\exists k \in \mathbb{Z}, n = 2k) \vee (\exists k \in \mathbb{Z}, n = 2k + 1)$ (by part (a)).
   Case 1: Assume $(\exists k \in \mathbb{Z}, n = 2k)$.
   Consider $k_0 \in \mathbb{Z}$ such that $n = 2k_0$ (by $\exists$E).
   Then $n^2 - n = (2k_0)^2 - 2k_0 = 2(2{k_0}^2 - k_0)$.
   Let $k = 2{k_0}^2 - k_0$.
   Then $k \in \mathbb{Z}$ (by closure of $\mathbb{Z}$ under $+, \times$).
   So $\exists k \in \mathbb{Z}, n^2 - n = 2k$ (by $\exists$I).
   Case 2: Assume $(\exists k \in \mathbb{Z}, n = 2k + 1)$.
   Consider $k_1 \in \mathbb{Z}$ such that $n = 2k_1 + 1$ (by $\exists$E).
   Then $n^2 - n = (2k_1 + 1)^2 - (2k_1 + 1) = 4{k_1}^2 + 4k_1 + 1 - 2k_1 - 1 = 2(2{k_1}^2 - k_1)$.
   Let $k = 2{k_1}^2 - k_1$.
   Then $k \in \mathbb{Z}$ (by closure of $\mathbb{Z}$ under $+, \times$).
   So $\exists k \in \mathbb{Z}, n^2 - n = 2k$ (by $\exists$I).
   Thus $\exists k \in \mathbb{Z}, n^2 - n = 2k$ (since we concluded it in all cases).
   Since n was an arbitrary element of $\mathbb{Z}$, $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 - n = 2k$ (by $\forall$I).

   Marking scheme for proofs (5 marks):

   - 2 marks: general appropriateness and clarity of the proof structure
   - 2 marks: overall content/correctness of the argument
   - 1 mark: any necessary justification of steps is provided

   (c) [6 marks] First we need to think about what we need to prove, and how to write it symbolically in a way we know we can prove it. We could write this statement symbolically as
   $\forall n \in \mathbb{Z}, \frac{n(n+1)}{2} \in \mathbb{Z}$
   or
   $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, \frac{n(n+1)}{2} = k$.
   The actual proof structure for these two alternatives is quite similar. We'll prove the latter statement.

   Assume $n \in Z$.
   Then n is either even or odd.
   That is, $(\exists k \in \mathbb{Z}, n = 2k) \vee (\exists k \in \mathbb{Z}, n = 2k + 1)$ (by part (a)).
   Case 1: Assume $(\exists k \in \mathbb{Z}, n = 2k)$.
   Consider $k_0 \in \mathbb{Z}$ such that $n = 2k_0$ (by $\exists$E).
   Then $\frac{n(n+1)}{2} = \frac{2k_0(n+1)}{2} = k_0(n + 1)$.
   Let $k = k_0(n + 1)$.
   Then $k \in \mathbb{Z}$ (by closure of $\mathbb{Z}$ under $+, \times$).
   So $\exists k \in \mathbb{Z}, \frac{n(n+1)}{2} = 2k$ (by $\exists$I).
   Case 2: Assume $(\exists k \in \mathbb{Z}, n = 2k + 1)$.

Consider $k_1 \in \mathbb{Z}$ such that $n = 2k_1 + 1$ (by $\exists$E).
Then $\frac{n(n+1)}{2} = \frac{n(2k_1+1+1)}{2} = \frac{2n(k_1+1)}{2} = n(k_1+1)$.
Let $k = n(k_1+1)$.
Then $k \in \mathbb{Z}$ (by closure of $\mathbb{Z}$ under $+, \times$).
So $\exists k \in \mathbb{Z}, \frac{n(n+1)}{2} = 2k$ (by $\exists$I).
Thus $\exists k \in \mathbb{Z}, \frac{n(n+1)}{2} = 2k$ (since we concluded it in all cases).
Since n was an arbitrary element of $\mathbb{Z}$, $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, \frac{n(n+1)}{2} = 2k$ (by $\forall$I).

2. [17 marks]

(a) [6 marks] The statement is false. We'll disprove it by proving the negation.

The negation is $\exists i \in \mathbb{N}, ((a_i > a_{i+1}) \vee (a_{i+1} > a_{i+2})) \wedge ((a_i < a_{i+1}) \vee (a_{i+1} < a_{i+2}))$. The proof is as follows.

Let $i = 2$. Then $i \in \mathbb{N}$.
Then $a_2 = 1, a_3 = 0, a_4 = 1$ ((by definition of (A))).
Then $(a_2 > a_3)$ and $(a_3 < a_4)$.
Then $(a_2 > a_3) \vee (a_3 > a_4)$ and $(a_2 < a_3) \vee (a_3 < a_4)$ ((by $\vee$I)).
Thus $((a_2 > a_3) \vee (a_3 > a_4)) \wedge ((a_2 < a_3) \vee (a_3 < a_4))$ ((by $\wedge$I)).
Thus $\exists i \in \mathbb{N}, ((a_i > a_{i+1}) \vee (a_{i+1} > a_{i+2})) \wedge ((a_i < a_{i+1}) \vee (a_{i+1} < a_{i+2}))$ ((by $\exists$I)).

(b) [5 marks] The statement is true.

Assume $i \in \mathbb{N}$.
Then $a_i \leq 1$.
And $a_{i+2} \leq 1$.
Also $a_{i+4} \leq 1$.
So $a_i + a_{i+2} + a_{i+4} \leq 3 \leq 4$.
Thus $\forall i \in \mathbb{N}, a_i + a_{i+2} + a_{i+4} \leq 4$. ((by $\forall$I))

(c) [6 marks] The statement is true.

Let $i = 2$.
Then $i \in \mathbb{N}$.
Assume $j \in \mathbb{N}$.
Assume $k \in \mathbb{N}$.
Assume $j = 3k$.
Then $a_j = a_{3k}$.
Also, k is a multiple of $2 \vee k$ is one more than a multiple of $4 \vee k$ is three more than a multiple of 4.
That is, $(\exists m \in \mathbb{N}, k = 2m) \vee (\exists m \in \mathbb{N}, k = 4m+1) \vee (\exists m \in \mathbb{N}, k = 4m+3)$.
Case 1: Assume $(\exists m \in \mathbb{N}, k = 2m)$.
Consider $m \in \mathbb{N}$ such that $k = 2m$. ((by $\exists$E))
Then $a_{3k} = a_{2 \cdot 3m} = 1$.
And $i + k = 2 + 2m = 2(m+1)$, so $a_{i+k} = 1 = a_j$.
Case 2: Assume $(\exists m \in \mathbb{N}, k = 4m+1)$.
Consider $m \in \mathbb{N}$ such that $k = 4m+1$. ((by $\exists$E))
Then $3k = 3(4m+1) = 4(3m)+3$, so $a_{3k} = 0$.
And $i + k = 2 + 4m + 1 = 4m + 3$, so $a_{i+k} = 0 = a_j$.
Case 3: Assume $(\exists m \in \mathbb{N}, k = 4m+3)$.
Consider $m \in \mathbb{N}$ such that $k = 4m+3$. ((by $\exists$E))
Then $3k = 3(4m+3) = 4(3m)+9 = 4(3m+2)+1$, so $a_{3k} = 1$.

And $i + k = 2 + 4m + 3 = 4(m + 1) + 1$, so $a_{i+k} = 1 = a_j$.
Thus $a_j = a_{i+k}$. ((since it was concluded in each case))
So $(j = 3k) \Rightarrow (a_j = a_{i+k})$. ((by $\Rightarrow$I))
So $\forall k \in \mathbb{N}, (j = 3k) \Rightarrow (a_j = a_{i+k})$. ((by $\forall$I))
So $\forall j \in \mathbb{N}, \forall k \in \mathbb{N}, (j = 3k) \Rightarrow (a_j = a_{i+k})$. ((by $\forall$I))
Thus $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, \forall k \in \mathbb{N}, (j = 3k) \Rightarrow (a_j = a_{i+k})$. ((by $\exists$I))

3. [20 marks]

(a) [8 marks] First we need to define our domains. Let I be the set of individuals (people), let P be the set of political parties, and let S be the set of subjects. Then:

(S1) $\forall x \in I, \forall y \in I, \forall p \in P, (x \neq y \wedge S(x, p) \wedge S(y, p)) \Rightarrow V(x, y)$

(S2) $\forall x \in I, \forall y \in I, V(x, y) \Rightarrow \exists s \in S, \neg A(x, y, s)$

(S3) $\exists x \in I, \exists y \in I, x \neq y \wedge \forall s \in S, A(x, y, s)$

(b) (i). [6 marks] "There is no party supported by all citizens."
$\neg \exists p \in P, \forall x \in I, S(x, p)$.

We will prove this statement by contradiction:

Assume (for contradiction) that $\exists p \in P, \forall x \in I, S(x, p)$.
Consider $p \in P$ such that $\forall x \in I, S(x, p)$ ((by $\exists$E)).
Consider $x, y \in I$ such that $x \neq y \wedge \forall s \in S, A(x, y, s)$ ((by $\exists$E on (S3))).
Then $x \neq y$ and $\forall s \in S, A(x, y, s)$ ((by $\wedge$E)).
So $S(x, p)$ and $S(y, p)$ ((by $\exists$E, line 2 and $x, y \in I$)).
Thus $x \neq y \wedge S(x, p) \wedge S(y, p)$ ((by $\wedge$I)).
But $(x \neq y \wedge S(x, p) \wedge S(y, p)) \Rightarrow V(x, y)$ ((by $\forall$E on (S1))).
So $V(x, y)$ ((by $\Rightarrow$E)).
Now $V(x, y) \Rightarrow \exists s \in S, \neg A(x, y, s)$ ((by $\forall$E on (S2))).
So $\exists s \in S, \neg A(x, y, s)$ ((by $\Rightarrow$E)).
We may rewrite this as $\neg \forall s \in S, A(x, y, s)$.
But on line 4 we knew $\forall s \in S, A(x, y, s)$.
Hence we have reached a contradiction!
Thus $\neg \exists p \in P, \forall x \in I, S(x, p)$ ((by $\neg$I)).

(ii). [6 marks] "If there is a party with a supporter, then there are at least two people."
$(\exists p \in P, \exists x \in I, S(x, p)) \Rightarrow (\exists x \in I, \exists y \in I, x \neq y)$.

The way we wrote (S3) makes this trivially easy:

Assume $\exists p \in P, \exists x \in I, S(x, p)$.
Consider $x, y \in I$ such that $x \neq y \wedge \forall s \in S, A(x, y, s)$ ((by $\exists$E on (S3))).
Then $x \neq y$ ((by $\wedge$E)).
So $\exists x \in I, \exists y \in I, x \neq y$ ((by $\exists$I)).
Thus $(\exists p \in P, \exists x \in I, S(x, p)) \Rightarrow (\exists x \in I, \exists y \in I, x \neq y)$ ((by $\Rightarrow$I)).