Due: By 12:00 noon on Thursday, December 6. **Worth**: 10%
You must complete and sign an assignment cover page, and attach it (with a staple) to the front of your assignment. Assignments should be handed into the drop box in BA 2220.

1. [10 marks] Maureen's sock drawer contains a number of white socks and a number of black socks. Each morning, she randomly takes two socks out of the drawer and checks whether they are the same colour. If they are not the same colour, she'll throw the black sock in a pile by the bed, return the white sock to the drawer, and will go sockless for the day. If the socks are the same colour, she'll wear them, but grab a black sock from the pile by the bed and return it to the drawer (there are always some black socks in the pile, if she ever runs out she steals some more from her roommate).

   Clearly she will eventually have only one sock in her sock drawer, since the total number of socks in the drawer decreases by one each day (in computerese, this is a *loop variant*). The question Maureen is curious about is what colour will the last sock in the drawer be? We'll answer this question below.

   (a) Suppose that there are an even number of white socks in the drawer at the beginning of the day. What can we say about the number of white socks in the drawer at the end of the day?

   (b) Using our structured form, prove your answer from part (a).

   (c) Suppose there was initially an even number of white socks in the drawer. What colour will the last sock be? Justify your answer.

   (d) Suppose there was initially an odd number of white socks in the drawer. What colour will the last sock be? Justify your answer.

2. [14 marks] Consider the following program, where A is an array of n integers:

```
b = A[0];
for (i = 0; i < A.length; i++) {
    if (A[i] >= b) {
        b = A[i];
        s = A[i];
        for (j = i-1; j >= 0; j--)
          s = s + A[j];
    }
}
```

   (a) Describe an array of length n that is a worst-case input for this algorithm (in terms of number of lines executed). Explain why your answer is a worst-case input.

   (b) Let $T(n)$ be the worst-case number of lines executed by this program over all arrays of length n. Prove that $T(n) \in \Theta(n^2)$.

3. [15 marks] Consider the normalized floating point system $\mathcal{F}$ with $\beta = 2$, $t = 7$, $e_{\max} = 8$, $e_{\min} = -7$ that uses round-to-nearest.

   (a) How many real numbers are representable exactly in $\mathcal{F}$? Justify your answer.

   (b) Give the decimal (base 10) representation of the largest and smallest positive real numbers representable in $\mathcal{F}$.

   (c) Give an example of a decimal number that will cause overflow in $\mathcal{F}$. Explain.

   (d) Give an example of a decimal number that will cause underflow in $\mathcal{F}$. Explain.

(e) Suppose a real number x is represented in $\mathcal{F}$ by $x' = 1.100101 \times 2^2$. What range of decimal values could x have been? Justify your answer.

4. [9 marks] Suppose we need to compute the expression $1 - (1 - x)^2$ for $x = 2.04$ in the normalized floating point system with $\beta = 10$, $t = 3$, $e_{\max} = 2$, $e_{\min} = -3$ that uses round-to-nearest.

   (a) Evaluate $1 - (1 - x)^2$ by performing the addition and multiplication operations in this system. Show your work.

   (b) Using the identity $(1 - x)^2 = 1 - 2x + x^2$ we note that $1 - (1 - x)^2 = 2x - x^2$. Evaluate the expression $2x - x^2$ in this system. Show your work.

   (c) Notice that we can also rewrite this expression as $x(2 - x)$. Evaluate $x(2 - x)$ in this system. Show your work.

   (d) Which computation is more stable for this value of x? Give a short explanation of why it is more stable.