This is a closed-book test: no books, no notes, no computers (of any kind), no calculators, no phones, etc. allowed. The only aid permitted is an 8.5 by 11 inch aid sheet. You may write (as small as you like) on both sides of the aid sheet, but you cannot have any “attachments” to the aid sheet.

Do NOT turn this page over until you are TOLD to start.

Duration of the test: 50 minutes (12:10 to 1:00 PM).

Write your answers to ALL questions in the test booklets provided.

Please fill-in ALL the information requested on the front cover of EACH test booklet that you use.

The test consists of two pages, including this one. Make sure you have both pages.

The test consists of 3 questions. Answer all 3 questions. The mark for each question is listed at the start of the question.

The test was written with the intention that you would have ample time to complete it. You will be rewarded for concise well-thought-out answers, rather than long rambling ones. We seek quality rather than quantity.

Moreover, an answer that contains relevant and correct information as well as irrelevant or incorrect information will be awarded fewer marks than one that contains the same relevant and correct information only.

Write legibly. Unreadable answers are worthless.
1. \[5 \text{ marks}\]

Consider the sequence \(x_k = 1/k!\) for \(k = 0, 1, 2, \ldots\). This sequence clearly converges to \(x^* = 0\) as \(k \to \infty\). Is the rate of convergence of \(x_k\) to \(x^*\) linear, super-linear or quadratic? Give the highest of these three rates that the sequence enjoys.

Justify your answer.

2. \[10 \text{ marks: 5 marks for each part}\]

Consider the function
\[f(x_1, x_2) = (x_1^2 + x_2)^2\]

At the point \(\hat{x} = (0, 1)^T\), consider the search direction \(p = (1, -1)^T\).

(a) Show that \(p\) is a descent direction for \(f\) at \(\hat{x}\).

(b) For \(\hat{x}\) and \(p\) fixed, find all the minimizers \(\alpha^*\) of \(f(\hat{x} + \alpha p)\).

3. \[5 \text{ marks}\]

Consider the steepest descent method with exact line searches applied to the quadratic function
\[f(x) = \frac{1}{2} x^T Ax - b^T x + c\]

where \(c \in \mathbb{R}\), \(x\) and \(b \in \mathbb{R}^n\), and \(A\) is an \(n \times n\) real symmetric positive-definite matrix.

We showed in class that \(x^* = A^{-1}b\) is the unique minimizer of \(f(x)\). You can use this result without proof.

Your textbook notes that the \(\alpha_n\) that minimizes the one-dimensional steepest-descent line-search-problem \(\phi(\alpha) = f(x_n - \alpha \nabla f(x_n))\) is

\[\alpha_n = \frac{\nabla f(x_n)^T \nabla f(x_n)}{\nabla f(x_n)^T A \nabla f(x_n)}\]

You can also use this result without proof.

Suppose that the initial guess \(x_0\) for the steepest descent method is such that \(x_0 - x^*\) is parallel to an eigenvector of \(A\). (That is, \(x_0 - x^* = \gamma v\) for some nonzero \(\gamma \in \mathbb{R}\) and some eigenvector \(v\) of \(A\).)

Show that the steepest descent method converges to \(x^*\) in one step. (That is, show that \(x_1 = x^*\), where \(x_1 = x_0 - \alpha_0 \nabla f(x_0)\) is the first iterate generated by the steepest descent method.)