This assignment is due at the **start** of your lecture on Friday, 26 January 2018.

The purpose of the assignment is to check that you are following what we’ve been talking about in class this week.

1. **[5 marks]**
   Show that, for any vector norm $\| \cdot \|$ for $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,
   $$|\|x\| - \|y\|| \leq \|x - y\|$$

2. **[5 marks]**
   Sometimes you see the *matrix norm subordinate to a vector norm* defined as
   $$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$ (1)
   and other times you see it defined as
   $$\|A\| = \max_{\|x\|=1} \|Ax\|$$ (2)
   Show that
   $$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$ (3)
   So definitions (1) and (2) are equivalent.

   Note that $A$ may be an $m \times n$ matrix, with $m \neq n$. In this case, the norm for $x$ in equations (1), (2) and (3) is a vector norm for $\mathbb{R}^n$, while the norm for $Ax$ in equations (1), (2) and (3) is a vector norm for $\mathbb{R}^m$. So these two norms are different. So, in your proof, don’t assume that the vector norm for $x$ is the same as the vector norm for $Ax$. However, the vector norm for $x$ is the same everywhere it occurs in equations (1), (2) and (3). Similarly, the vector norm for $Ax$ is the same everywhere it occurs in equations (1), (2) and (3).

   Your proof should hold for any vector norm for $\mathbb{R}^m$ and any vector norm for $\mathbb{R}^n$.

3. **[5 marks]**
   Show that, for any matrix $A \in \mathbb{R}^{m \times n}$ and any vector $x \in \mathbb{R}^n$,
   $$\|Ax\| \leq \|A\| \|x\|$$
   where $\|A\|$ is the matrix norm for $A$ defined by (1) or equivalently (2) and $\|Ax\|$ and $\|x\|$ are the vector norms for $Ax$ and $x$ used in (1) and (2).
4. [5 marks]

Show that there is always an \( x^* \neq 0 \) such that

\[
\|Ax^*\| = \|A\| \|x^*\|
\]  

(4)

where again \( \|A\| \) is the matrix norm for \( A \) defined by (1) or equivalently (2) and \( \|Ax\| \) and \( \|x\| \) are the vector norms for \( Ax \) and \( x \) used in (1) and (2).

Moreover, show that, if \( x^* \) satisfies (4), then \( \hat{x} = \alpha x^* \) satisfies

\[
\|A\hat{x}\| = \|A\| \|\hat{x}\|
\]

for all \( \alpha \in \mathbb{R} \).

Hence, there are infinitely many \( x^* \) that satisfy (4).

5. [5 marks]

Show that an isolated local minimizer of a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a strict local minimizer of \( f \).

(See pages 13 and 14 of your textbook for the definitions of isolated local minimizer and strict local minimizer of \( f \).)