Other topics.

1. Preconditioned CG (NW, p. 118)

Let \( \hat{x} = Cx \) where \( C \) is non-singular.

\[
\Phi(\hat{x}) = \Phi(C^{-1}\hat{x}) = \frac{1}{2} \hat{x}^T (C^{-T}AC^{-1}) \hat{x} - (C^{-T}b)^T \hat{x}
\]

Solve \( \hat{x} = C^{-T}b \)

Possible advantage: \( K(C^{-T}AC^{-1}) \ll K(A) \)

E.g. Suppose \( A = LL^T \) (Cholesky factorization)

Let \( C = L^T \)

Then \( (C^{-T})AC^{-1} = L^{-1}LL^T L^{-T} = I \)

\( \Rightarrow K(C^{-T}AC^{-1}) = 1 \)

Incomplete Cholesky Factorization \( LL^T = A + E \) often \( K(L^{-1}AL^{-T}) \ll K(A) \)

Can implement precondition CG quite efficiently

See Alg. 5.3 on p. 119 (One extra solve with \( M \))
Matrix Free CG

CG needs matrix multiply $A p$

Suppose $A = \nabla^2 f(x_k)$

Then $A p = \nabla^2 f(x_k) p \approx \frac{\nabla f(x_k + h p) - \nabla f(x_k)}{h}$

Can have problems with accuracy.

Can also use automatic differentiation to compute $\nabla^2 f(x_k) p$
**Inexact Newton Method**  \( (N+\nu)\) p. 165

**Newton step**  
\[ p_k = -\left( \nabla^2 f(x_k) \right)^{-1} \nabla f(x_k) \]

\[ \Rightarrow \quad \nabla^2 f(x_k) p_k = -\nabla f(x_k) \]

\[ \Rightarrow \quad \nabla^2 f(x_k) p_k + \nabla f(x_k) = 0 \]

**Inexact Newton methods** - use an iterative method (e.g. CG) to compute a \( p_k \) that satisfies

\[ \| \nabla^2 f(x_k) p_k + \nabla f(x_k) \| \leq \eta_k \| \nabla f(x_k) \| \]

- \( \eta_k \leq \eta < 1 \) get linear convergence \( (\text{Thm 7.1}) \)
- \( \eta_k \to 0 \) get super linear convergence
- \( \eta_k = O\left(\| \nabla f(x_k) \| \right) \) get quadratic convergence

**Trust-Region - Newton-CG**  \( (N+\nu), \) p. 170

- Similar to Dog-Leg. Stop when approx to \( p_k \) leaves the Trust Region

\[ p_k = p_k - \nabla^2 f(x_k)^{-1} \nabla f(x_k) \]

\[ \text{approx to } p_k \text{ generated by CG.} \]
Case 1 \[ \| P_k \| \leq \Delta \]

Stop when
\[ \| \nabla^2 f(x_k) \tilde{P}_k + \nabla f(x_k) \| \leq \eta_k \| \nabla f(x_k) \| \]

Case 2 \[ \| P_k \| > \Delta \]

Let \[ \| P_k^{(0)} \| \leq \Delta \] and \[ \| P_k^{(\infty)} \| > \Delta \]

Consider line \[ P_k^{(0)} + z (P_k^{(\infty)} - P_k^{(0)}) \]
\[ z \in [0, 1] \]

Choose \[ z^* \] s.t. \[ \tilde{P}_k = P_k^{(0)} + z^* (P_k^{(\infty)} - P_k^{(0)}) \]
Satisfies \[ \| \tilde{P}_k \| = \Delta \]
Like dogleg method

(a) $|p_k^{(i)}|$ increasing with $i$

(b) $\phi(x_k + p_k^{(i)})$ decreasing with $i$.