Do **NOT** turn this page over until you are **TOLD** to start.

Please fill-in **ALL** the information requested on the front cover of **EACH** exam booklet that you use.

The exam consists of 5 pages, including this one. Make sure you have all 5 pages.

The exam consists of 4 questions. **Answer all 4 questions.** The mark for each question is listed at the start of the question. Do the questions that you feel are easiest first.

**Note:** students must receive a mark of at least 35% on the exam in order to pass the course.

The exam was written with the intention that you would have ample time to complete it. You will be rewarded for concise well-thought-out answers, rather than long rambling ones. **We seek quality rather than quantity.**

Moreover, an answer that contains relevant and correct information as well as irrelevant or incorrect information will be awarded fewer marks than one that contains the same relevant and correct information only.

**Write legibly. Unreadable answers are worthless.**
You may find the following definitions useful.

The shift operator \( \mathcal{E} z(x) = z(x + h) \)

The forward difference operator \( \Delta_+ z(x) = z(x + h) - z(x) \)

The backward difference operator \( \Delta_- z(x) = z(x) - z(x - h) \)

The central difference operator \( \Delta_0 z(x) = z(x + h/2) - z(x - h/2) \)

The averaging operator \( \Upsilon_0 z(x) = \frac{1}{2} [z(x + h/2) + z(x - h/2)] \)

1. [10 marks]

We showed in class that, for every \( s \geq 1 \),

\[
\frac{1}{h^s} \left( \Delta_+^s - \frac{s}{2} \Delta_+^{s+1} \right) z(x) = z^{(s)}(x) - h^2 \frac{s(3s + 5)}{24} z^{(s+2)}(x) + \mathcal{O}(h^3)
\]

as \( h \to 0 \), where \( z^{(s)}(x) \) is the \( s \)th derivative of \( z \) with respect to \( x \) and \( z^{(s+2)}(x) \) is the \( (s+2) \)rd derivative of \( z \) with respect to \( x \). (This is problem 8.3 on page 167 of your textbook.)

Similarly, for every \( s \geq 1 \), there is a constant \( a_s \) (which depends on \( s \)) and a constant \( c_s \neq 0 \) (which also depends on \( s \)) such that

\[
\frac{1}{h^s} \left( \Delta_-^s + a_s \Delta_-^{s+1} \right) z(x) = z^{(s)}(x) + h^2 c_s z^{(s+2)}(x) + \mathcal{O}(h^3)
\]

as \( h \to 0 \), where, as above, \( z^{(s)}(x) \) is the \( s \)th derivative of \( z \) with respect to \( x \) and \( z^{(s+2)}(x) \) is the \( (s+2) \)rd derivative of \( z \) with respect to \( x \).

Determine \( a_s \) and \( c_s \) for all \( s \geq 1 \).

Show all your work.

[Comment: the difference expressions on the left sides of equations (1) and (2) are both second-order approximations to \( z^{(s)}(x) \) and the principal error constant is \( -s(3s + 5)/24 \) for (1) and \( c_s \) for (2).]
2. [10 marks: 5 marks for each part]

Consider the Poisson equation

\[
\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = f(x, y) \quad \text{for} \quad (x, y) \in \Omega_L \tag{3}
\]

in two-dimensions with Dirichlet boundary conditions

\[
u(x, y) = g(x, y) \quad \text{for} \quad (x, y) \in \partial \Omega_L \tag{4}
\]

where the domain \( \Omega_L \) is the \( L \)-shaped region

\[
\Omega_L = R_1 \cup R_2
\]

and \( R_1 \) and \( R_2 \) are the rectangles

\[
R_1 = \{(x, y) : -1 < x < 1 \text{ and } -1 < y < 0\}
\]

\[
R_2 = \{(x, y) : -1 < x < 0 \text{ and } 0 \leq y < 1\}
\]

Another way of describing \( \Omega_L \) is that it is the square

\[
S_1 = \{(x, y) : -1 < x < 1 \text{ and } -1 < y < 1\}
\]

with the smaller square

\[
S_2 = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}
\]

removed. That is,

\[
\Omega_L = \{(x, y) : (x, y) \in S_1 \text{ and } (x, y) \notin S_2\}
\]

For any integer \( N \geq 1 \), let \( h = 1/(N + 1) \) and consider the discretization

\[
x_i = -1 + i h \quad \text{for} \quad i = 0, 1, \ldots, 2(N + 1)
\]

\[
y_j = -1 + j h \quad \text{for} \quad j = 0, 1, \ldots, 2(N + 1)
\]

Note \((x_i, y_j) \in \Omega_L\) if either

(a) \( i \in \{1, 2, \ldots, 2N + 1\} \) and \( j \in \{1, 2, \ldots, N\} \)

(b) \( i \in \{1, 2, \ldots, N\} \) and \( j \in \{N + 1, N + 2, \ldots, 2N + 1\} \).

Using this discretization and the 5-point approximation to the Laplacian

\[
\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}
\]

where \( u_{i,j} \approx u(x_i, y_j) \), you can construct a system of linear equations

\[
A\hat{u} = \hat{f} + \hat{g} \tag{5}
\]

where \( \hat{u} \) is a vectorized version of \( \{u_{i,j} : (x_i, y_j) \in \Omega_L\} \), \( \hat{f} \) is a vectorized version of the function \( f(x, y) \) on the right side of the Poisson equation (3), and \( \hat{g} \) is a vector containing the boundary conditions corresponding to (4).

(a) Describe how to initialize the matrix \( A \) and the vectors \( \hat{f} \) and \( \hat{g} \) in (5).

Your description should be detailed enough so that a programmer, who doesn’t know anything about numerical methods for PDEs, can write a program to initialize the matrix \( A \) and the vectors \( \hat{f} \) and \( \hat{g} \).

(b) Show that the matrix \( A \) in (5) is nonsingular.
3. [10 marks: 5 marks for each part]

In Assignment 3, you used the Ritz-Galerkin method with the tensor product of piecewise linear hat (i.e., chapeau) basis functions to solve the two-dimensional Poisson equation

\[-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 32x(1 - x) + 32y(1 - y) \quad \text{for } x \in (0, 1) \text{ and } y \in (0, 1)\]

with Dirichlet boundary conditions \( u(x, y) = 0 \) if \( x = 0, x = 1, y = 0 \) or \( y = 1 \).

To be more explicit, let \( m \) be a positive integer and set \( h = 1/(m+1) \), \( x_i = i \cdot h \) for \( i = 1, \ldots, m \) and \( y_j = j \cdot h \) for \( j = 1, \ldots, m \). Then, for \( k = 1, 2, \ldots, m \), define the piecewise linear hat (i.e., chapeau) basis function in the \( x \)-direction by

\[
\varphi_k(x) = \begin{cases} 
\frac{x - x_{k-1}}{x_k - x_{k-1}} & \text{for } x \in [x_{k-1}, x_k) \\
\frac{x_k - x_{k+1}}{x_{k+1} - x_k} & \text{for } x \in (x_k, x_{k+1}] \\
0 & \text{otherwise}
\end{cases}
\]

Define, \( \varphi_l(y) \), the piecewise linear hat (i.e., chapeau) basis function in the \( y \)-direction, similarly.

Then, for \( k = 1, 2, \ldots, m \) and \( l = 1, 2, \ldots, m \), set \( \varphi_{k,l}(x, y) = \varphi_k(x)\varphi_l(y) \).

The approximate solution generated by the Ritz-Galerkin method has the form

\[ u_m(x, y) = \sum_{k=1}^{m} \sum_{l=1}^{m} \gamma_{k,l} \varphi_{k,l}(x, y) \]

where \( \varphi_{k,l}(x, y) \) is the two-dimensional hat (i.e., chapeau) basis function described above and the coefficients \( \gamma_{k,l} \) are determined by solving the Galerkin equations which can be written in matrix form as

\[ A\hat{\gamma} = b \]

where \( \hat{\gamma} \) is a vectorized version of \( \{ \gamma_{k,l} : k = 1, 2, \ldots, m \text{ and } l = 1, 2, \ldots, m \} \).

(a) Describe how to initialize the matrix \( A \) in (6).

Your description should be detailed enough so that a programmer, who doesn’t know anything about numerical methods for PDEs, can write a program to initialize the matrix \( A \).

(b) Show that the matrix \( A \) in (6) is nonsingular.
4. [10 marks]
Assume $A$ is a symmetric positive definite matrix. Show that, for any starting guess $x^{(0)}$ and for any relaxation parameter $\omega \in (0, 2)$, the SOR iteration

$$(D + \omega L)x^{(k+1)} = ((1 - \omega)D - \omega U)x^{(k)} + \omega b$$

converges to the solution $x$ of $Ax = b$, where $A = D + L + U$ with

- $D$ a diagonal matrix (i.e. $D_{i,j} = 0$ if $i \neq j$),
- $L$ a strictly lower triangular matrix (i.e. $L_{i,j} = 0$ if $i \leq j$),
- $U$ a strictly upper triangular matrix (i.e. $U_{i,j} = 0$ if $i \geq j$).