This assignment is due at the start of our lecture on Monday, 13 March 2017.

For the questions that require you to write a MatLab program, hand in the program and its output as well as any written answers requested in the question. Your program should conform to the usual CS standards for comments, good programming style, etc. Try to format your output from your program so that it is easy for your TA to read your results.

Before writing your MatLab programs, you might find it useful to read the MatLab documentation on our course webpage http://www.cs.toronto.edu/~krj/courses/446/. You should use sparse matrices as much as possible. Read help in MatLab on sparfun, sparse and spdiags. After you have initialized a matrix using the MatLab sparse matrix routines, you can solve a system $Ax = b$ in MatLab by $x = A \backslash b$. MatLab will use an efficient sparse matrix factorization to solve the system.

Everyone who has registered for this course should have an account on CS Teaching Labs Computer System (i.e., formerly called the CDF System). You should be able to access the system remotely over the internet. There is more information about accessing your account at the start of Assignment 1. If you have any trouble with this, let me know and I will try to help.

Throughout this assignment, I refer to MatLab, but you can use Octave or one of the other MatLab clones instead. (See “MatLab Clones” on the course webpage.) However, MatLab clones are not 100% compatible with MatLab. So, run each of your final programs through MatLab to make sure that your program really runs under MatLab, since, when your TA marks your program, he will want to see a working MatLab program.

1. [5 marks]
   Do question 9.2 on page 202 of your textbook.

2. [10 marks]
   Use our corrected version of the Geršgorin criterion (Lemma 8.3 on page 157 of your textbook) to solve question 9.3 on page 202 of your textbook.

   State any additional assumptions that you need to ensure that the system is nonsingular.

   Recall that we already proved in class that this system is nonsingular by showing that the associated matrix is symmetric positive-definite, hence nonsingular. The point of this question is to use the Geršgorin criterion to give an alternative proof of this result.
3. [5 marks]
Do question 9.4 on page 202 of your textbook.
The definition of *positive definite* is given on page 185 of your textbook.
In this case, if the domain associated with the differential operator $L$ is $[c, d]$, then a function $v(x)$ satisfies the Dirichlet boundary conditions if

$$
v(c) = v'(c) = v(d) = v'(d) = 0$$

That is, both the function and its first derivative are zero at the two endpoints of the domain of $L$.

4. [15 marks]
Do question 9.7 on pages 202 and 203 of your textbook.
This question proves the remark following equation (9.12) on page 179 of your textbook that $\inf_{v \in \mathbb{H}} J(v) > -\infty$.
Note that all functions $v \in \mathbb{H}$ satisfy the boundary conditions $v(0) = \alpha$ and $v(1) = \beta$ as well as

$$
\int_0^1 [v(x)]^2 \, dx < \infty \quad \text{and} \quad \int_0^1 [v'(x)]^2 \, dx < \infty
$$

Moreover,

$$
\tilde{a}(v, v) = \int_0^1 \{ a(x)[v'(x)]^2 + b(x)[v(x)]^2 \} \, dx
$$

where $a(x) > 0$ and $b(x) \geq 0$ for all $x \in [0, 1]$. In addition,

$$
\langle f, v \rangle = \int_0^1 f(x) \, v(x) \, dx
$$

and

$$
\| v \| = \sqrt{\langle v, v \rangle}
$$

where

$$
\langle v, v \rangle = \int_0^1 [v(x)]^2 \, dx
$$

Moreover, you can assume that $a(x)$, $b(x)$ and $f(x)$ are “well-behaved” functions and that all the integrals that you need in this problem are well-defined and finite.
5. [10 marks]

Consider the two-point boundary value problem
\[ -y''(x) + \lambda^2 y(x) = x, \quad \text{for } x \in (0, 1), \]
\[ y(0) = y(1) = 1 \]

It is easy to verify that the solution to this problem is
\[ y(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + x/\lambda^2 \]

where
\[ c_2 = \frac{e^\lambda + 1/\lambda^2 - 1}{e^\lambda - e^{-\lambda}} \]
\[ c_1 = 1 - c_2 \]

In this assignment, use \( \lambda = \sqrt{10} \) (i.e., \( \lambda^2 = 10 \)).

Write a MatLab program that uses the Ritz-Galerkin method with piecewise linear chapeau (i.e., hat) basis functions (defined on page 176 of your textbook) on an equally spaced grid to solve the two-point boundary value problem (1) with \( \lambda^2 = 10 \). That is, let the gridpoints be \( x_i = ih \) for \( i = 0, 1, \ldots, m + 1 \) and \( h = 1/(m + 1) \), where \( m \) is an integer. (See below for the choices of \( m \).)

The approximate solution generated by the Ritz-Galerkin method has the form
\[ y_m(x) = \varphi_0(x) + \sum_{k=1}^{m} \gamma_k \varphi_k(x) \]

where \( \varphi_0(x) \) is any simple function that satisfies the boundary conditions associated with the problem (1) (i.e., \( \varphi_0(0) = \varphi_0(1) = 1 \)), \( \varphi_k(x) \) is the chapeau basis function defined on page 176 of your textbook and the \( \gamma_k \) are determined by solving the Galerkin equations (see (9.7) on page 174 of your textbook).

For each of \( m = 9, 19, 39, 79, 159, 319, 639 \), use your program to compute the Ritz-Galerkin solution \( y_m(x) \) to the two-point boundary value problem (1) with \( \lambda^2 = 10 \).

The maximum error in the numerical solution at the gridpoints \( \{x_i : i = 1, \ldots, m\} \),
\[ \max \{ |y(x_i) - y_m(x_i)| : i = 1, \ldots, m \} \]

is a good approximation to the infinity norm of the error in the numerical solution
\[ \|y - y_m\|_\infty = \max \{ |y(x) - y_m(x)| : x \in [0, 1] \} \]

Compute and print the maximum error in the numerical solution at the gridpoints (2) for \( m = 9, 19, 39, 79, 159, 319, 639 \).

How does this error decrease with \( h = 1/(m + 1) \)?

(Note that, for \( m = 9, 19, 39, 79, 159, 319, 639 \), \( m + 1 = 10, 20, 40, 80, 160, 320, 640 \) and \( h = 1/(m + 1) = 1/10, 1/20, 1/40, 1/80, 1/160, 1/320, 1/640 \). So, \( h \) is halved for each successive \( m \).)