Tutorial notes for week 7

Problem 1: Consider the search and decision versions of the problem **Minimum Subset Sum Problem** given as follows:

MSSS

Instance: A sequence of weights $\langle w_1, \ldots, w_d, t \rangle$, where t is the target amount (all weights and the target are positive integers given in binary).

Output: A subset $S \subseteq \{1, \ldots, d\}$ such that $\sum_{i \in S} w_i = t$ and $|S|$ is as small as possible, if such a subset exists; return "no" otherwise.

MSSD

Instance: A sequence of weights $\langle w_1, \ldots, w_d, t, B \rangle$, with the bound B (again, all positive integers, presented in binary).

Question: Is there a subset $S \subseteq \{1, \ldots, d\}$ such that $\sum_{i \in S} w_i = t$ and $|S| \leq B$?

Show that MSSS$\xrightarrow{p}$ MSSD. Note that to solve MSSS your algorithm should find an optimal set S, not just $|S|$. Also show that MSSD$\in NP$.

Solution: We show that MSSS$\xrightarrow{p}$ MSSD by giving a polytime algorithm for solving MSSS which has access to a solver for MSSD. Our algorithm has two parts: in the first part we compute the smalles $|S|$ such that $\sum_{i \in S} = t$ (if it exists). We can do this using linear search on $B = 1, 2, \ldots, d$, i.e. by quering MSSD($\langle w_1, \ldots, w_d, t, B \rangle$) with $B = 1$, then with $B = 2$, etc., and we stop the first time we get a YES answer. Let the result be B_{min}. If we never get a YES answer, then no such set exists, so we stop and return NO. (Why can we get away with linear search, rather than binary search?)

AFter part one is complete, we have B_{min} (or we know that no such set exists, in which case we stopped). In the second part we want to compute the memebers of a set S, where $|S| = B_{min}$. Again, we query MSSD weight by weight as follows:

- if MSSD($\langle w_2, \ldots, w_d, t, B_{min} \rangle$) is NO, then we know that weight $w_!$ is necessary, we output it, and consider w_2 with MSSD($\langle w_1, w_3, \ldots, w_d, t, B_{min} \rangle$).
- If it is YES, we know that w_1 is *not* necessary, so we go on to consider MSSD($\langle w_2, \ldots, w_d, t, B_{min} \rangle$).

To prove that MSSD$\in NP$: A certificate for this problem is a set $S \subseteq \{1, \ldots d\}$. It is easy to see that we can check the following two things in polytime: (i) $|S| \leq B$, and (ii) $\sum_{i \in S} w_i = t$.