Lipschitz Continuous O.D.E.s are Polynomial-Space Complete

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Problem

Given $g\colon [0,1] \times \mathbf{R} \to \mathbf{R}$, consider the IVP

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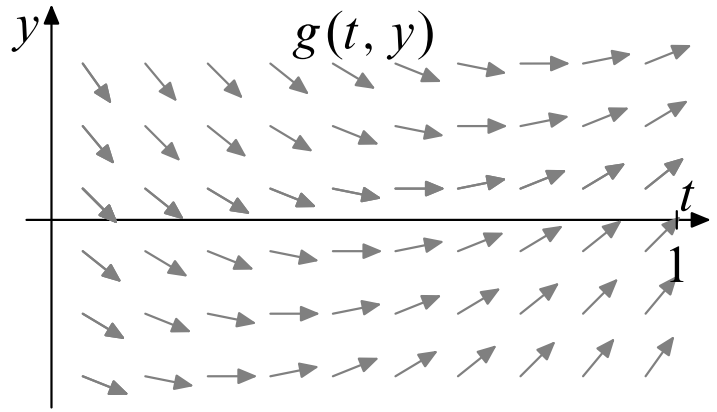

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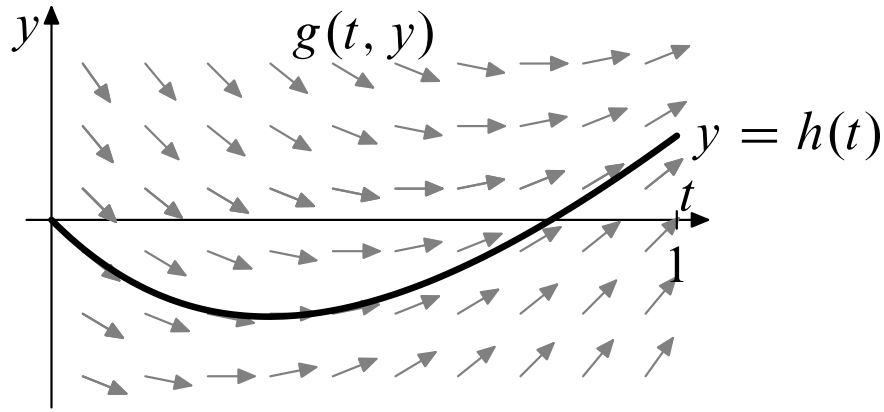

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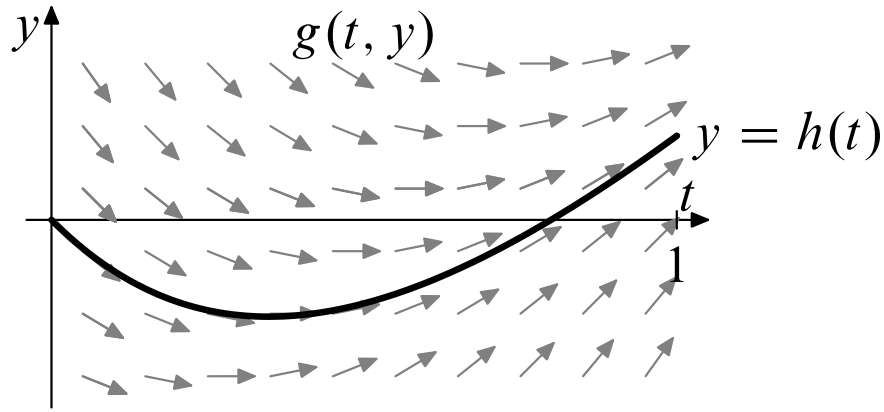

A sufficient condition for unique solution h is that

$$|g(t, y_0) - g(t, y_1)| \leq L|y_0 - y_1|, \qquad t \in [0,1],\ y_0, y_1 \in \mathbf{R}$$

for some $L \geq 0$ (Lipschitz continuity).

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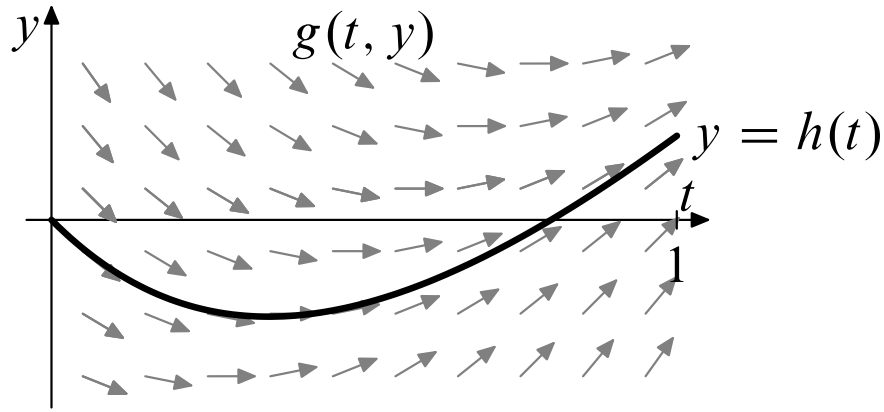

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Question

g is easy $\Longrightarrow$ h is easy? (in the sense of Computable Analysis)

Outline

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

1. Complexity of real functions
   How we define PTIME real functions
2. Warm-up: Integration
   Assuming g is PTIME and ignores the second argument, how complex can h be?
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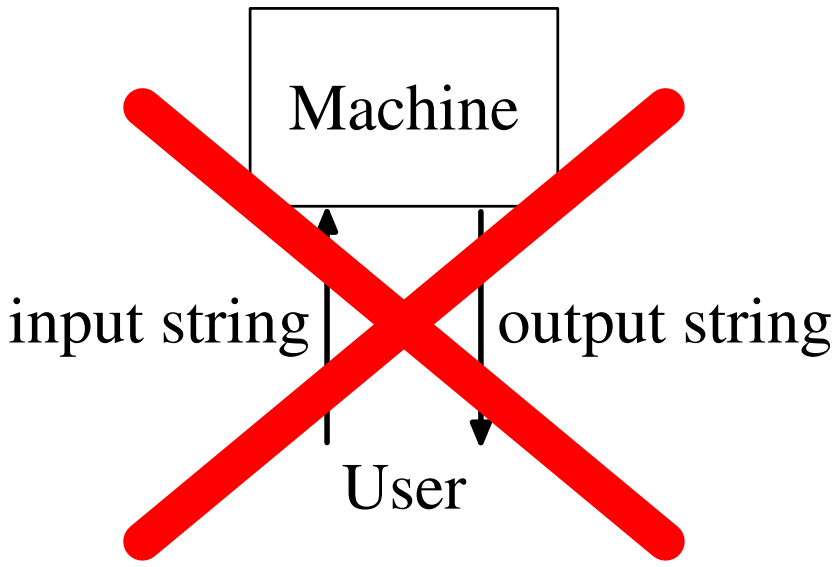
Read/write real numbers?

A real number cannot be encoded into a string.

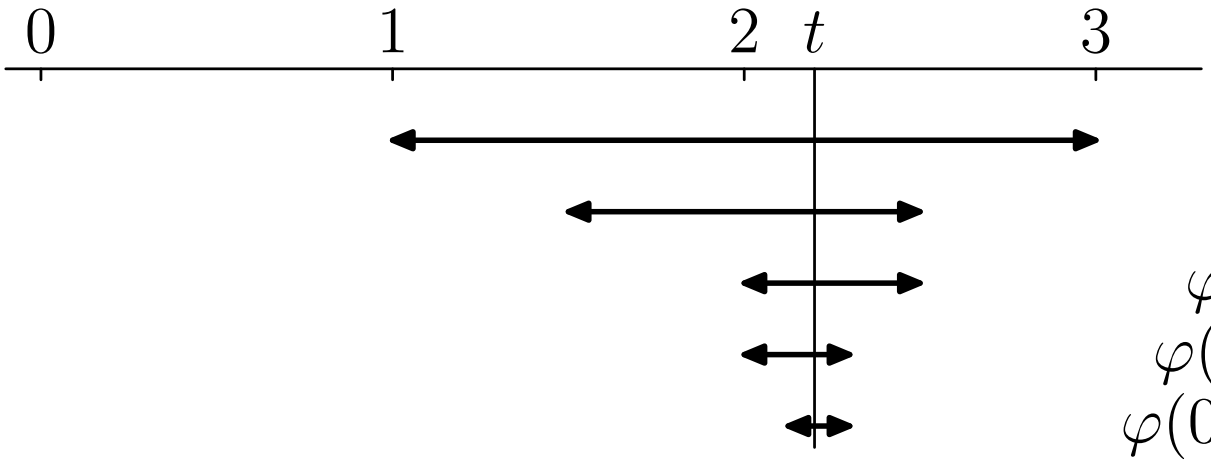


So we encode it into a function from strings to strings,
and give it to the machine as oracles.

Representation of real numbers

Definition

We say that $\varphi\colon \Sigma^* \to \Sigma^*$ is a name of $t \in \mathbf{R}$ if for each $n \in \mathbf{N}$, $\varphi(0^n)$ is (the binary expansion of) t rounded up or down at the nth bit below the point.



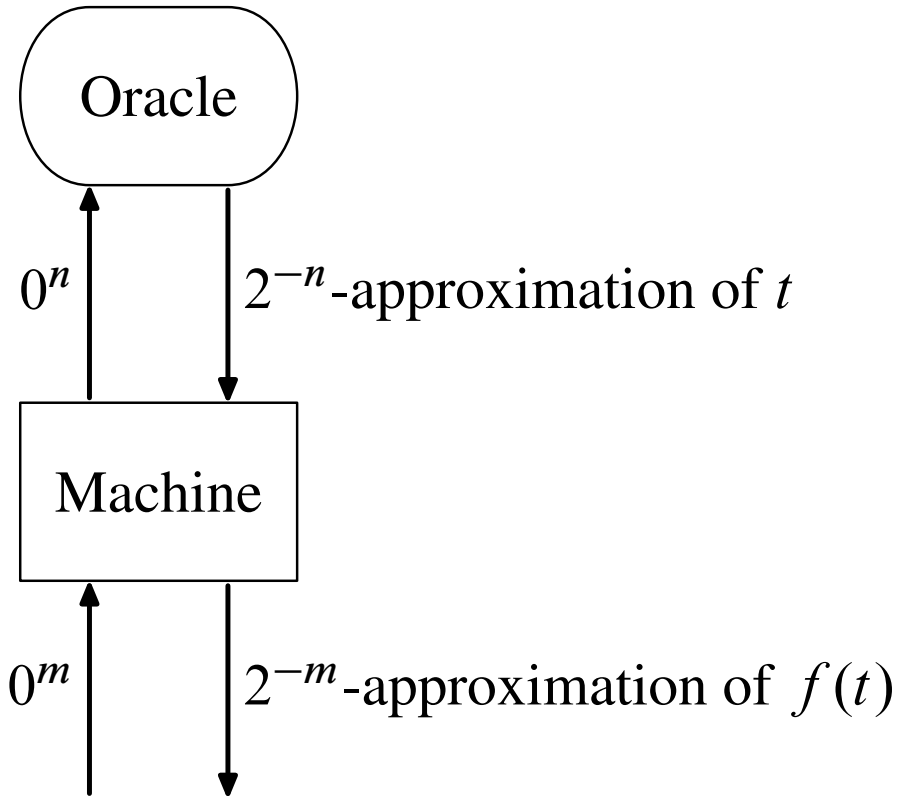
$$\begin{aligned}
\varphi(\varepsilon) &= +10. \\
\varphi(0) &= +10.0 \\
\varphi(00) &= +10.01 \\
\varphi(000) &= +10.001 \\
\varphi(0000) &= +10.0011 \\
&\vdots
\end{aligned}$$

(Each $\varphi(0^n)$ approximates t with precision 2^{-n})

Computing real functions

Definition (Grzegorzcyk 1955)

Machine M computes $f : [0,1] \to \mathbf{R}$ if, for any name φ of any $t \in [0,1]$, M^{φ} computes a name of $f(t)$.



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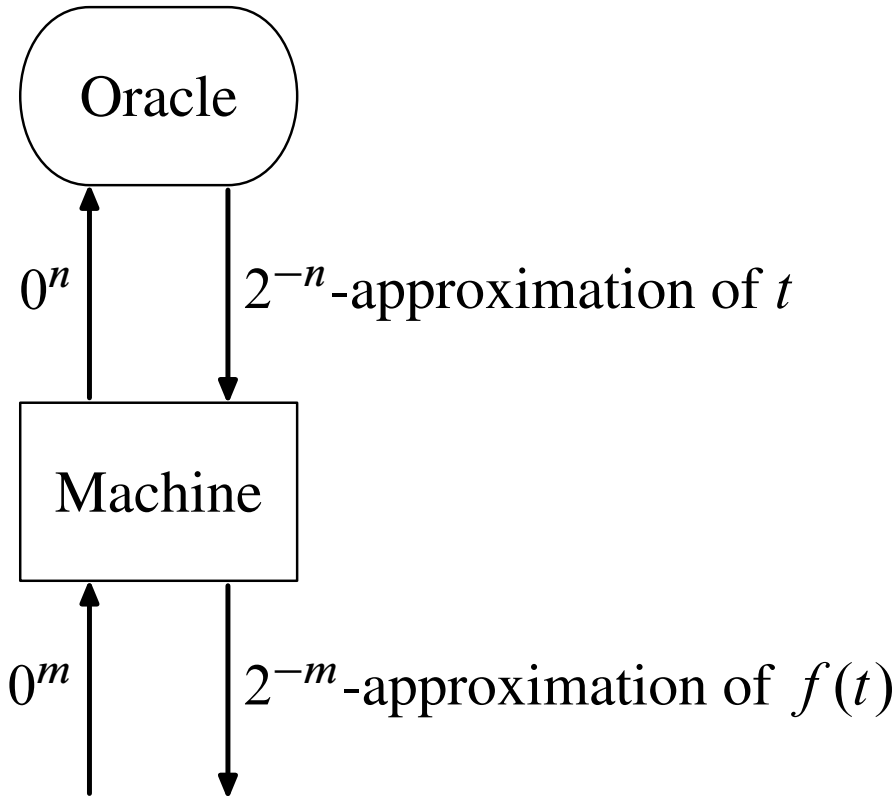

In other words,

- M Turing-reduces $f(t)$ to t.
- Given access to approximations of t to any precision, the machine yields $f(t)$ to any precision.

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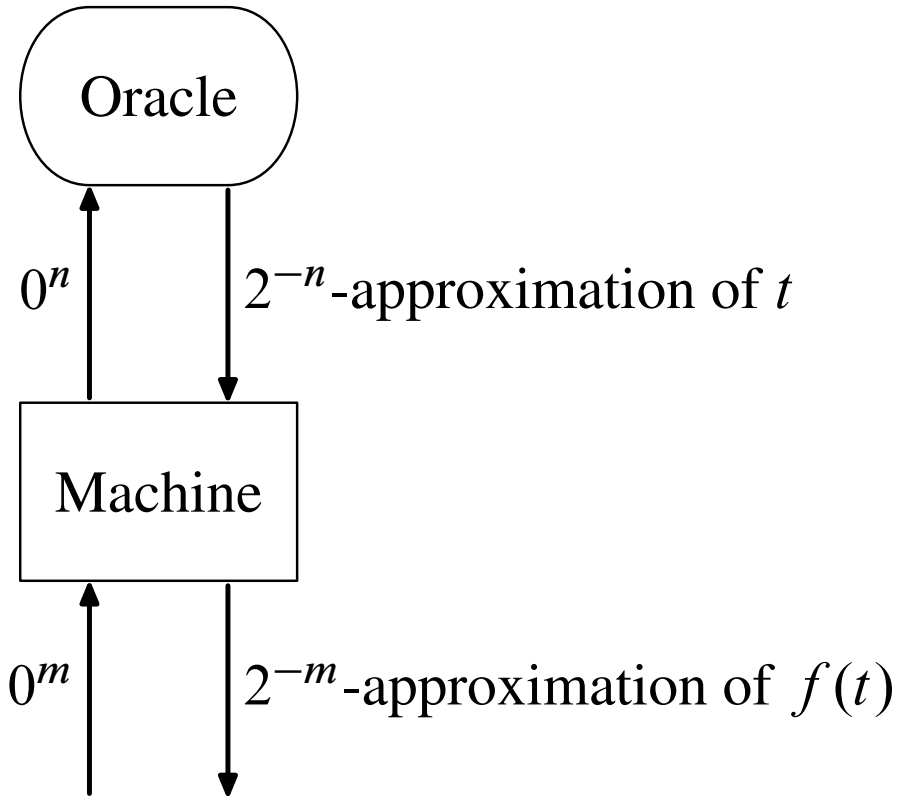

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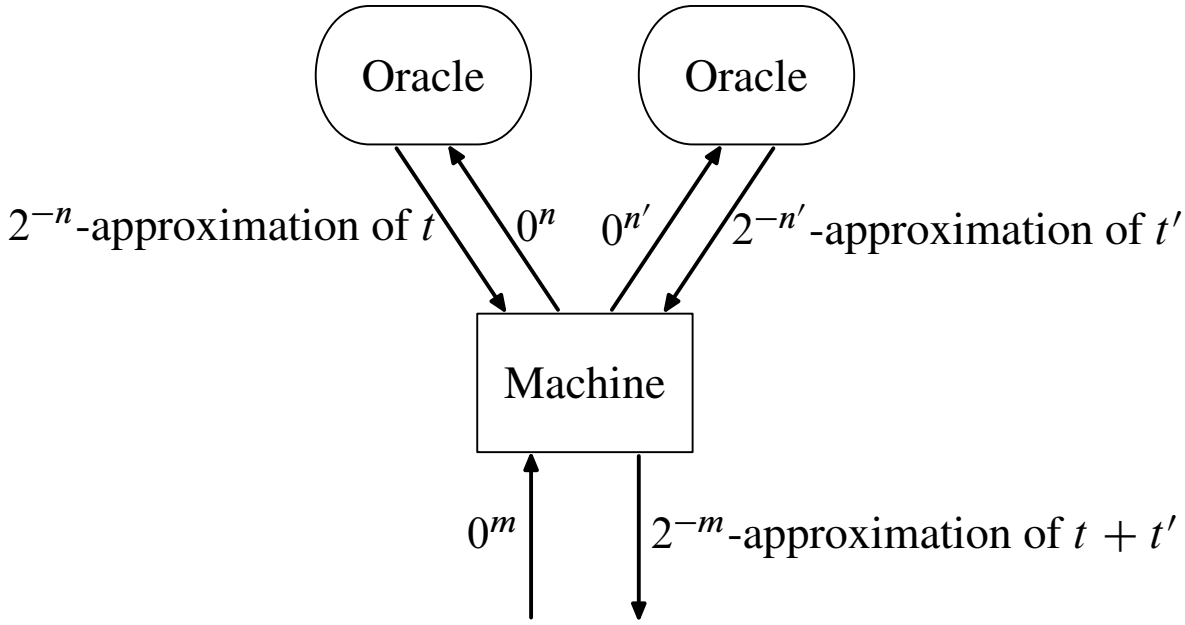
Note: Saying that M is PTIME means that it halts in time poly(m). In particular, $n \leq$ poly(m).

Example 1

Addition $+\colon [0,1] \times [0,1] \to \mathbf{R}$ is PTIME.

To do: Given (names of) t and t' as oracles and 0^m as input, output a 2^{-m}-approximation of $t + t'$.



- Ask the oracles for 2^{-m-2}-approximations of t and t'.
- Add the answers (as rational numbers).
- Output the closest rational number to the sum that has m bits below the point.

Example 2

$\exp\colon [0,1] \to \mathbf{R}$ is PTIME.

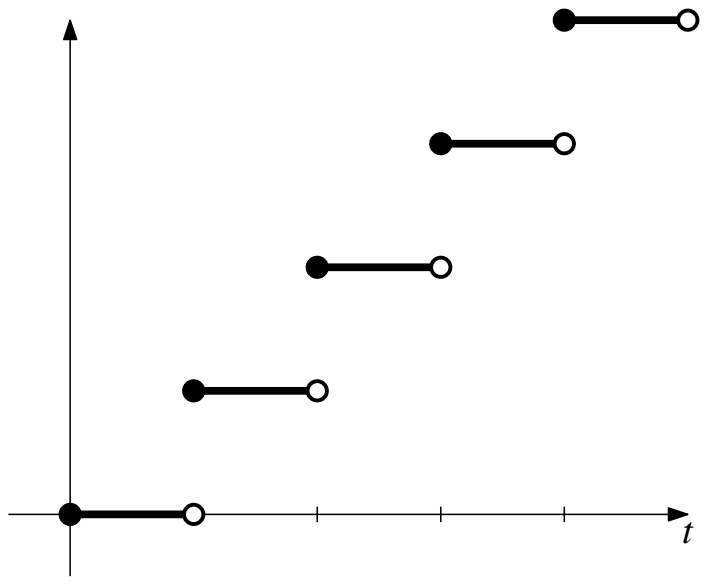
To get a 2^{-m}-approximation of the value

$$\exp t = \frac{1}{0!} + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots,$$

it suffices to compute a $2^{-m}/2$-approximation of the sum of the first m terms (because the remaining terms add up to at most $2^{-m}/2$).

Example 3

$\lfloor \cdot \rfloor : [0, 5] \longrightarrow \mathbf{R}$ is not computable.



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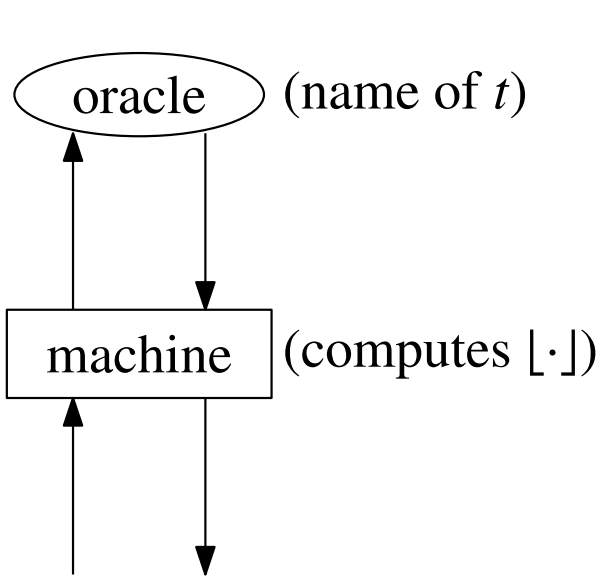

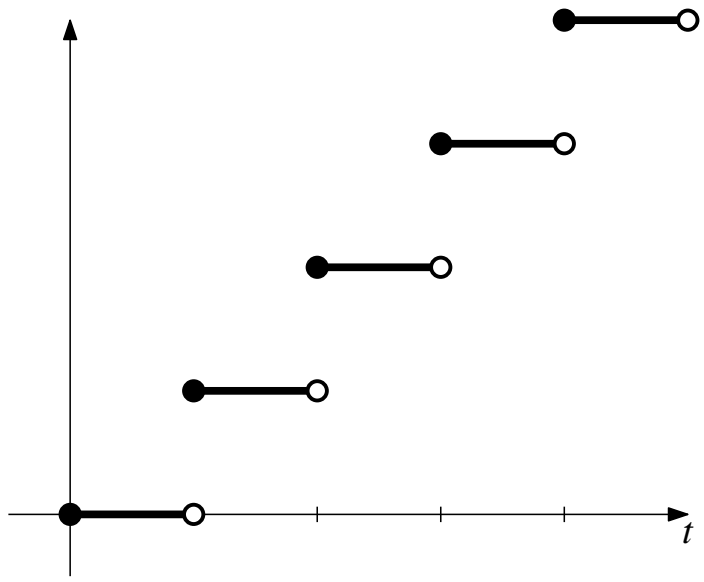

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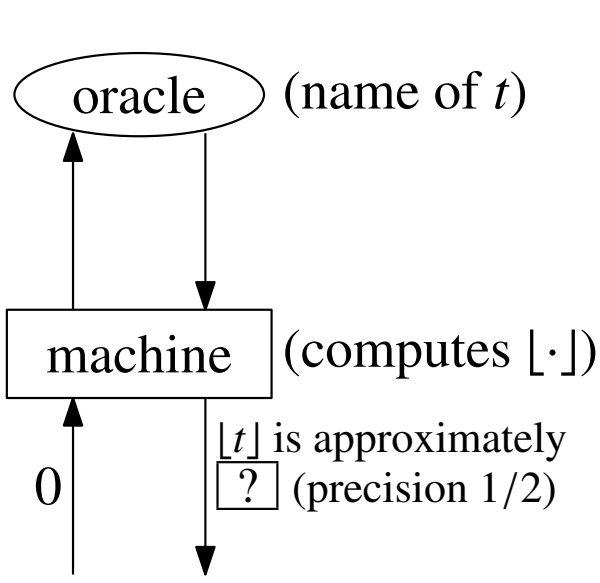

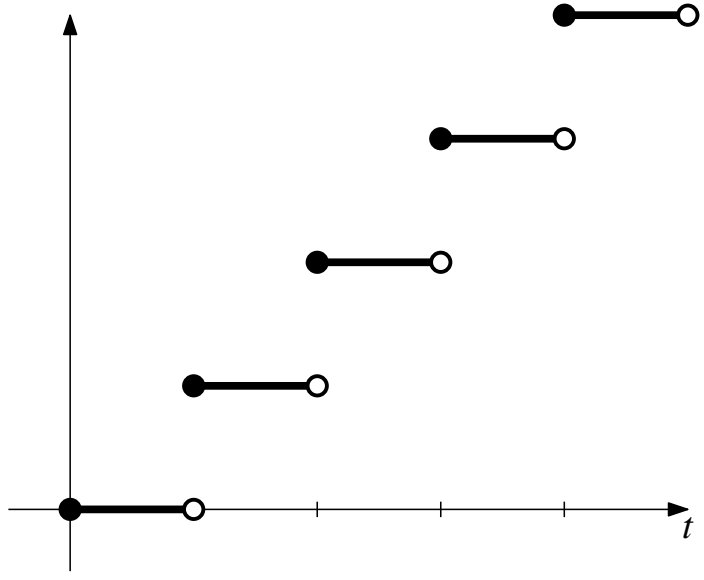

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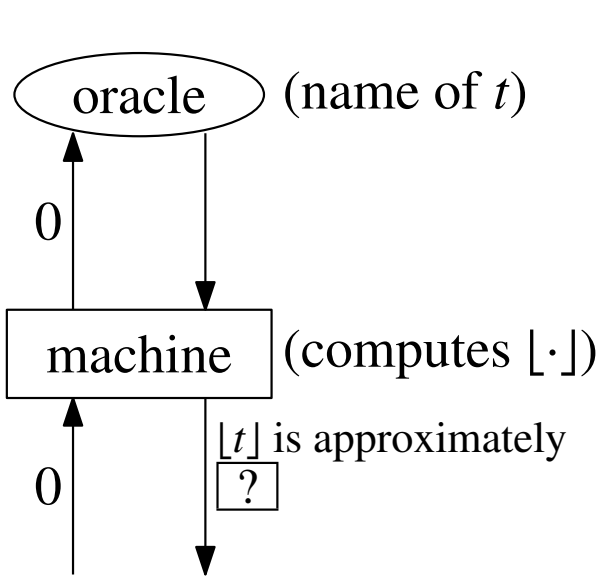

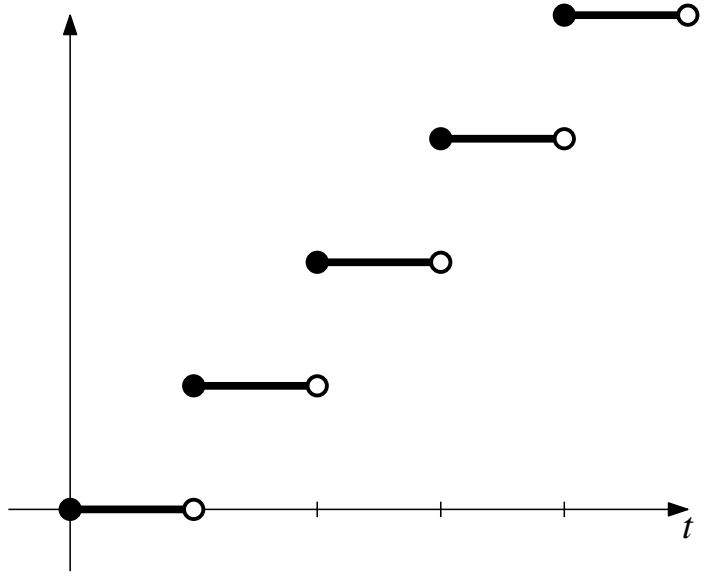

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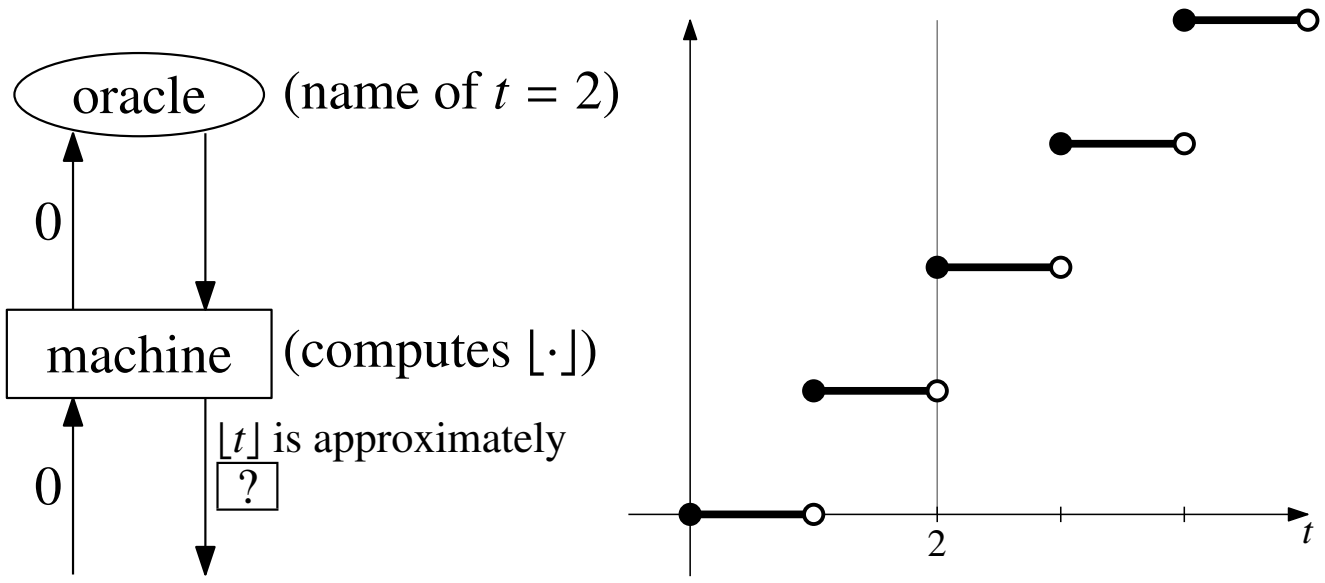

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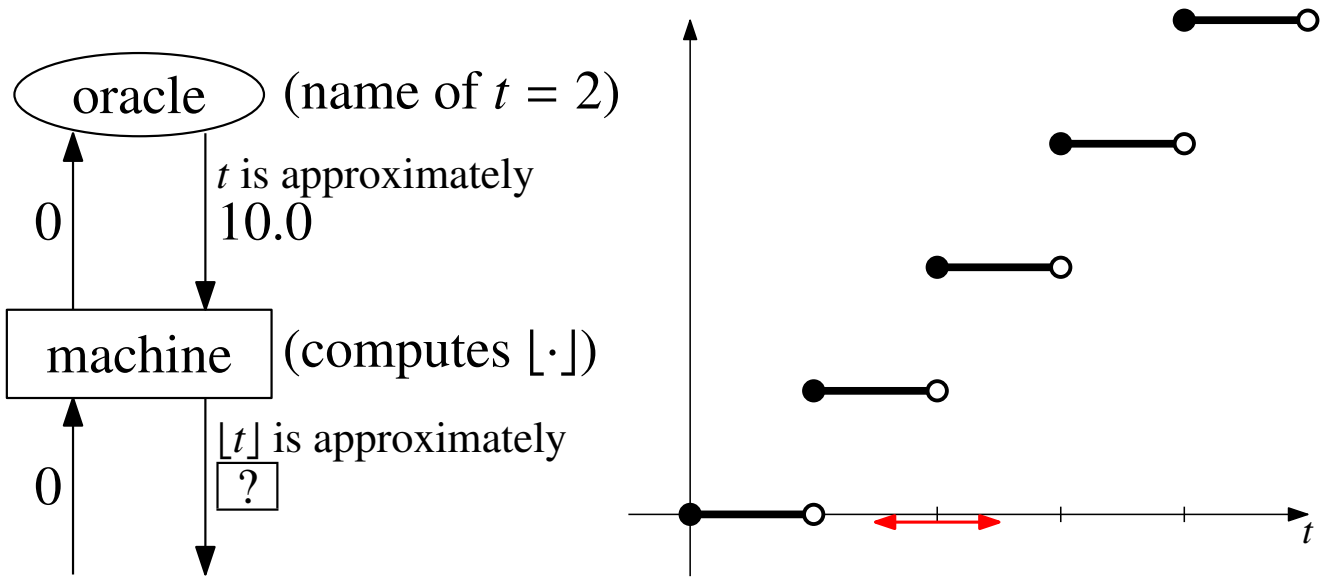

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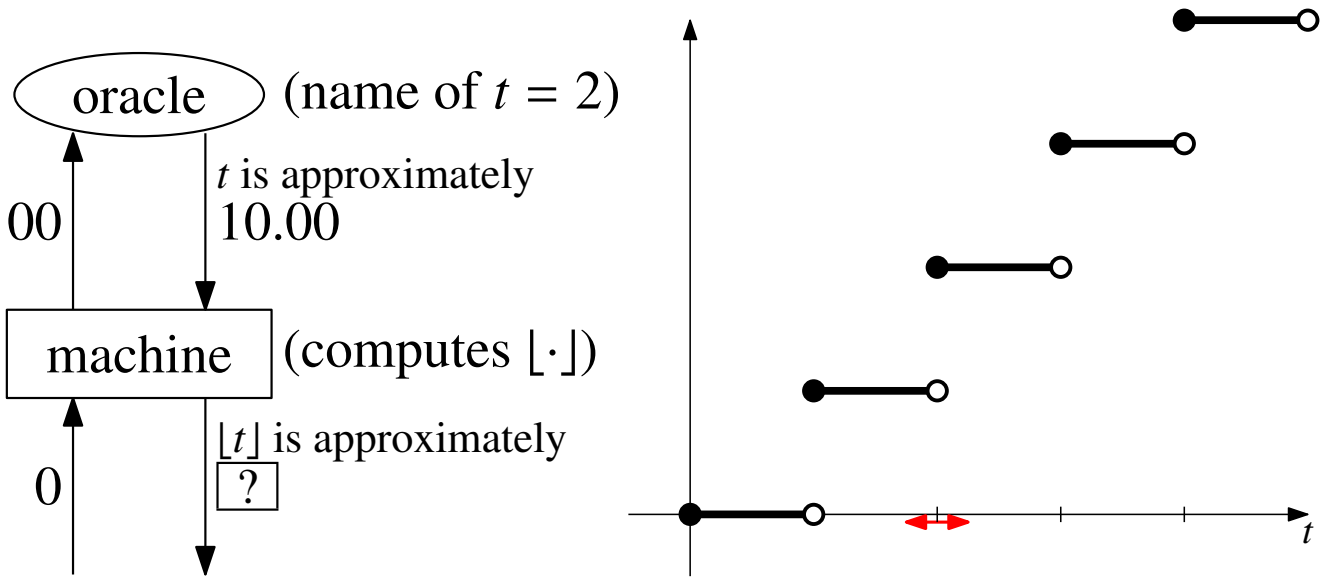

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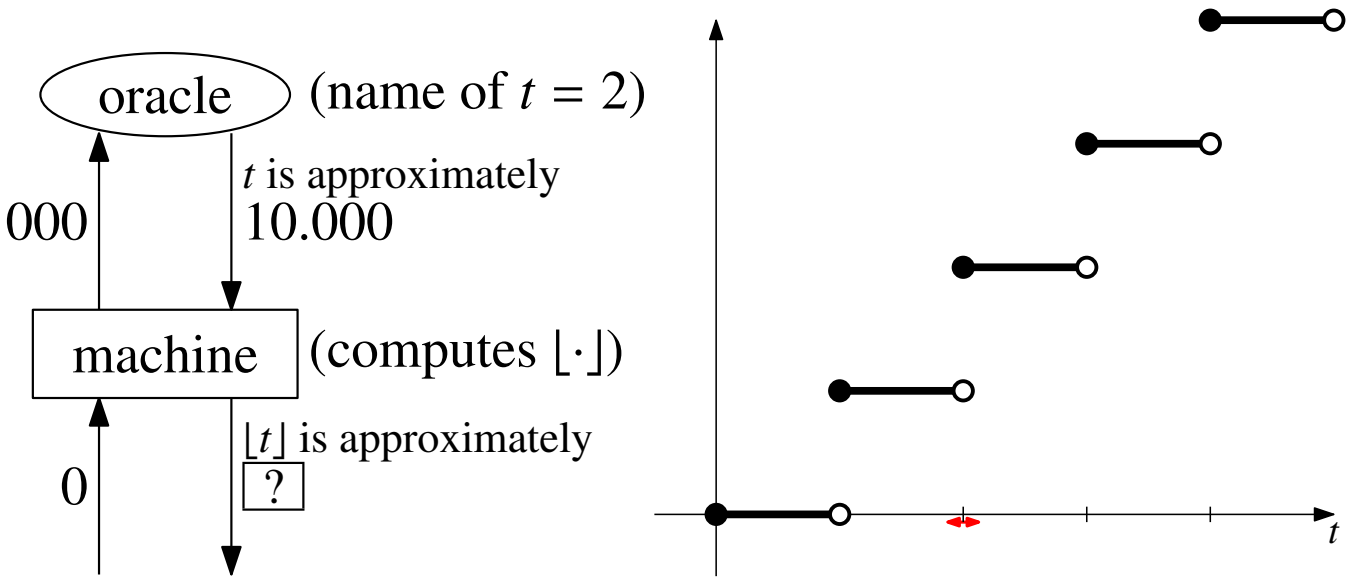

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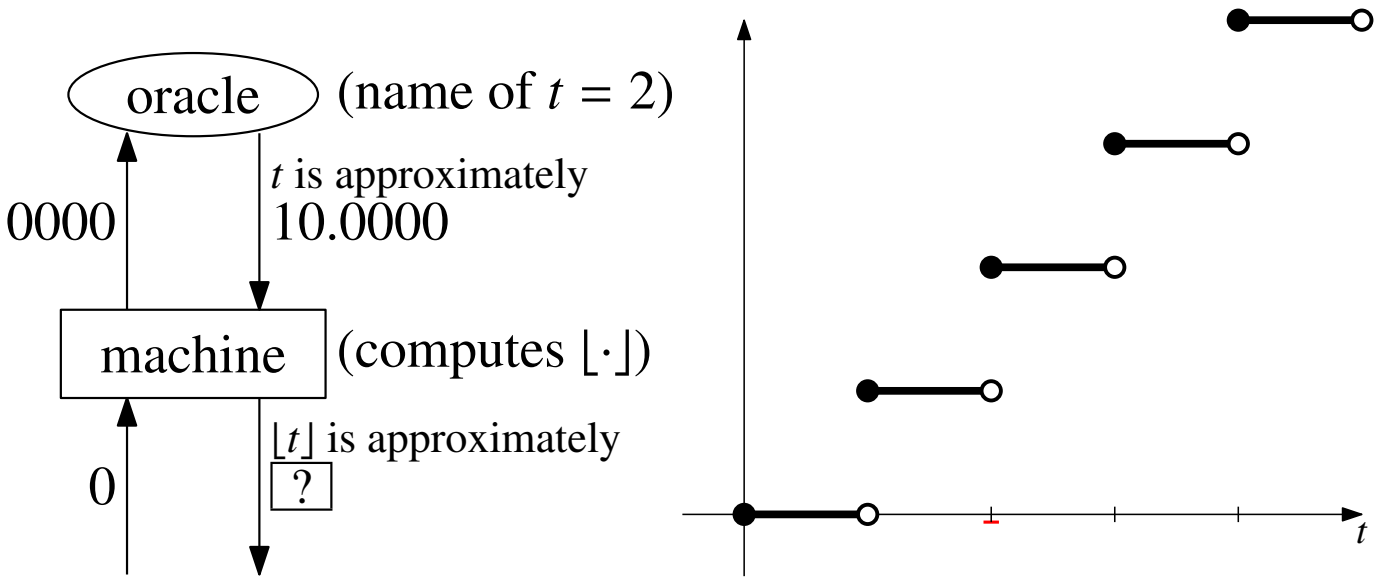


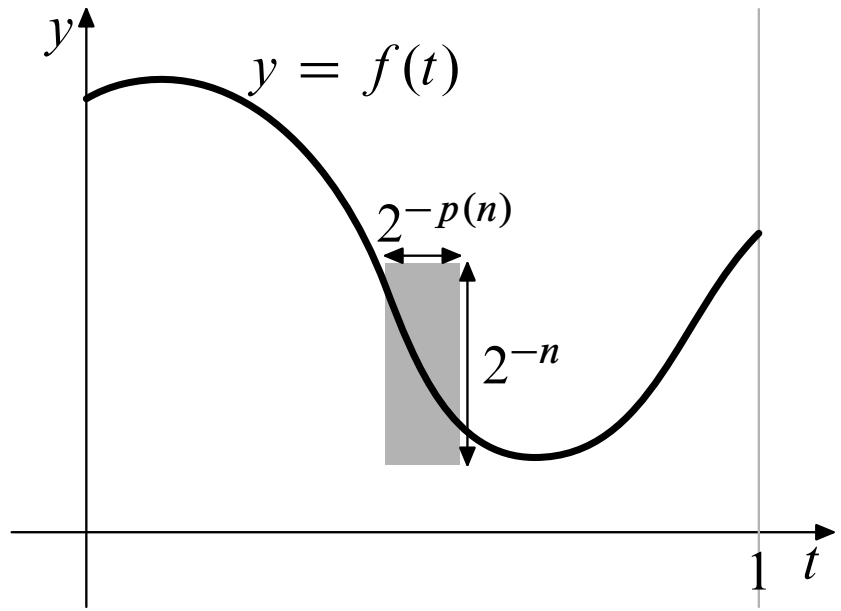
Computable functions are continuous

Theorem

- A computable real function is continuous.
- A PTIME real function has a polynomial modulus of continuity.

Modulus of continuity p:

$$|t - t'| < 2^{-p(n)} \implies |f(t) - f(t')| < 2^{-n}.$$



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Theorem (essentially [Friedman 1984])

There are $g\colon [0,1] \to \mathbf{R}$ and $h\colon [0,1] \to \mathbf{R}$ such that

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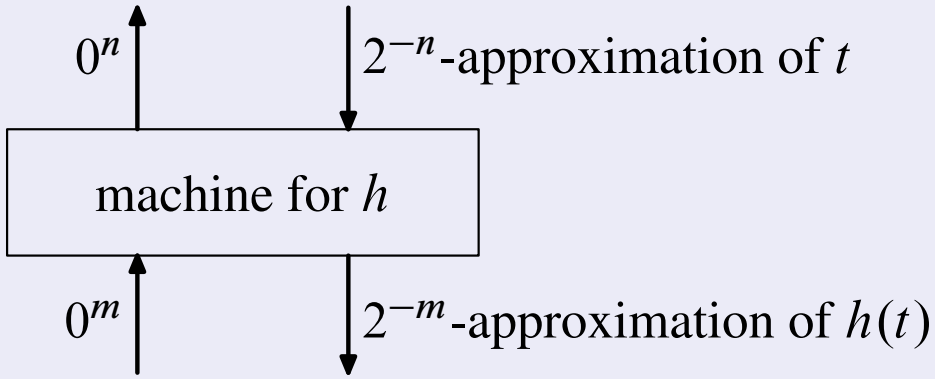

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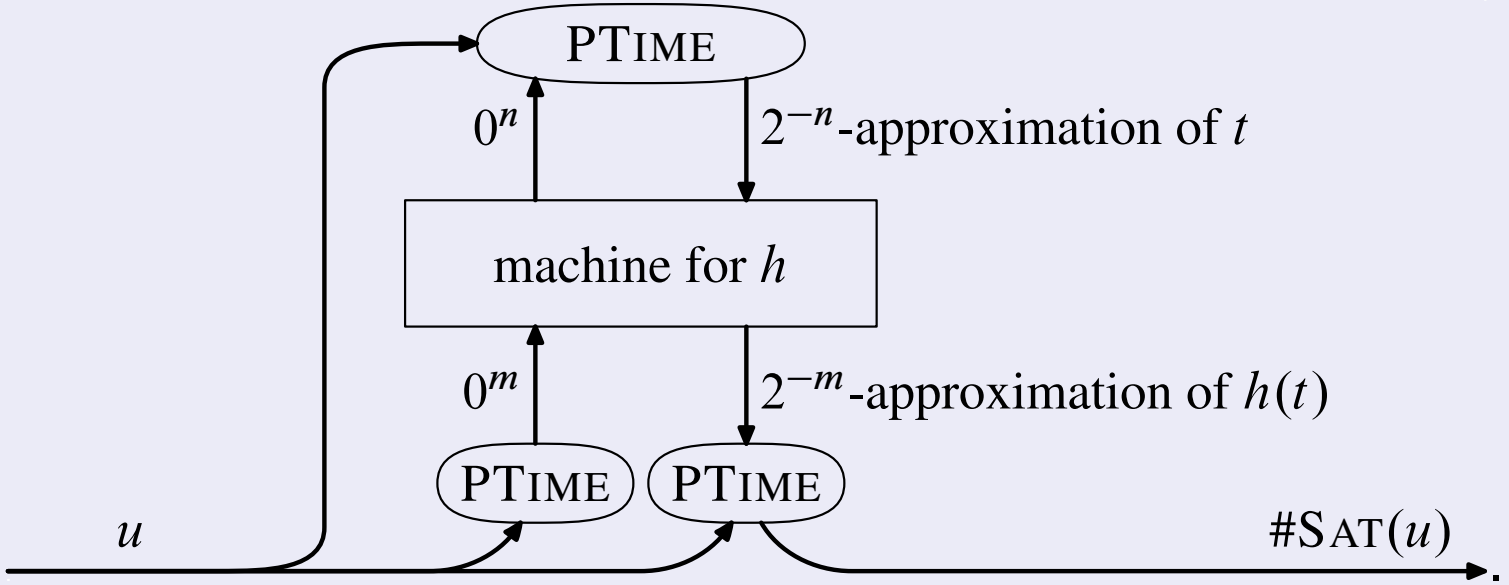
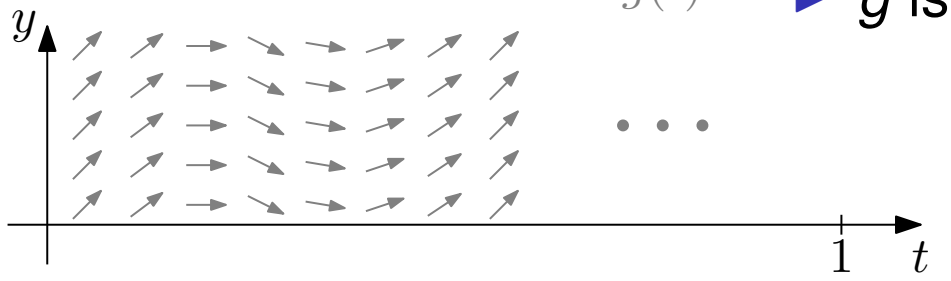


Reducing #SAT to integration



What we want:

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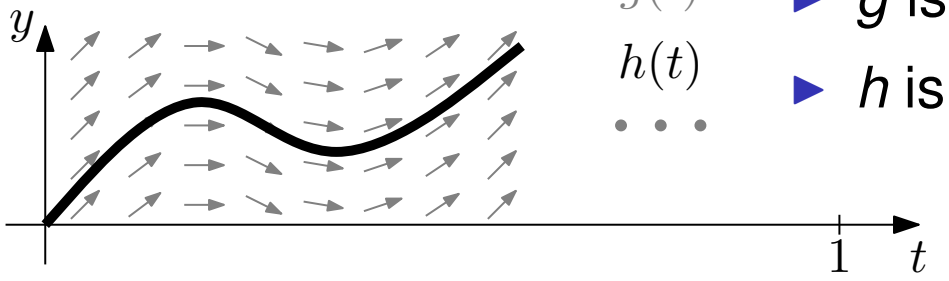

$g(t)$

$h(t)$

• • •

What we want:

- g is easy (PTIME);
- h is hard (#SAT reduces to it).

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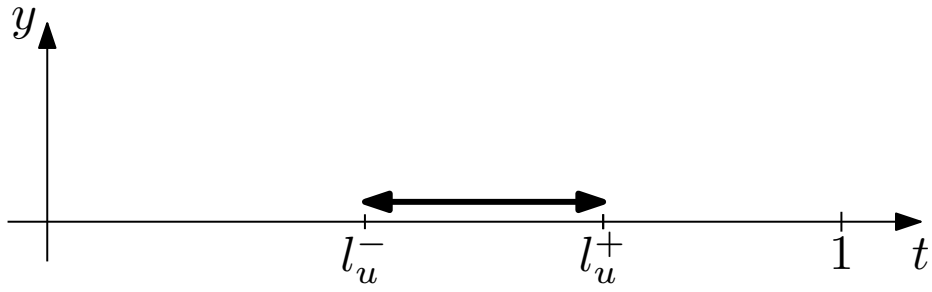

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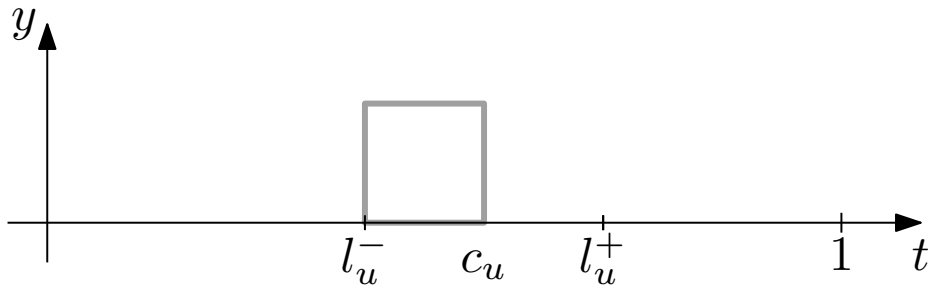

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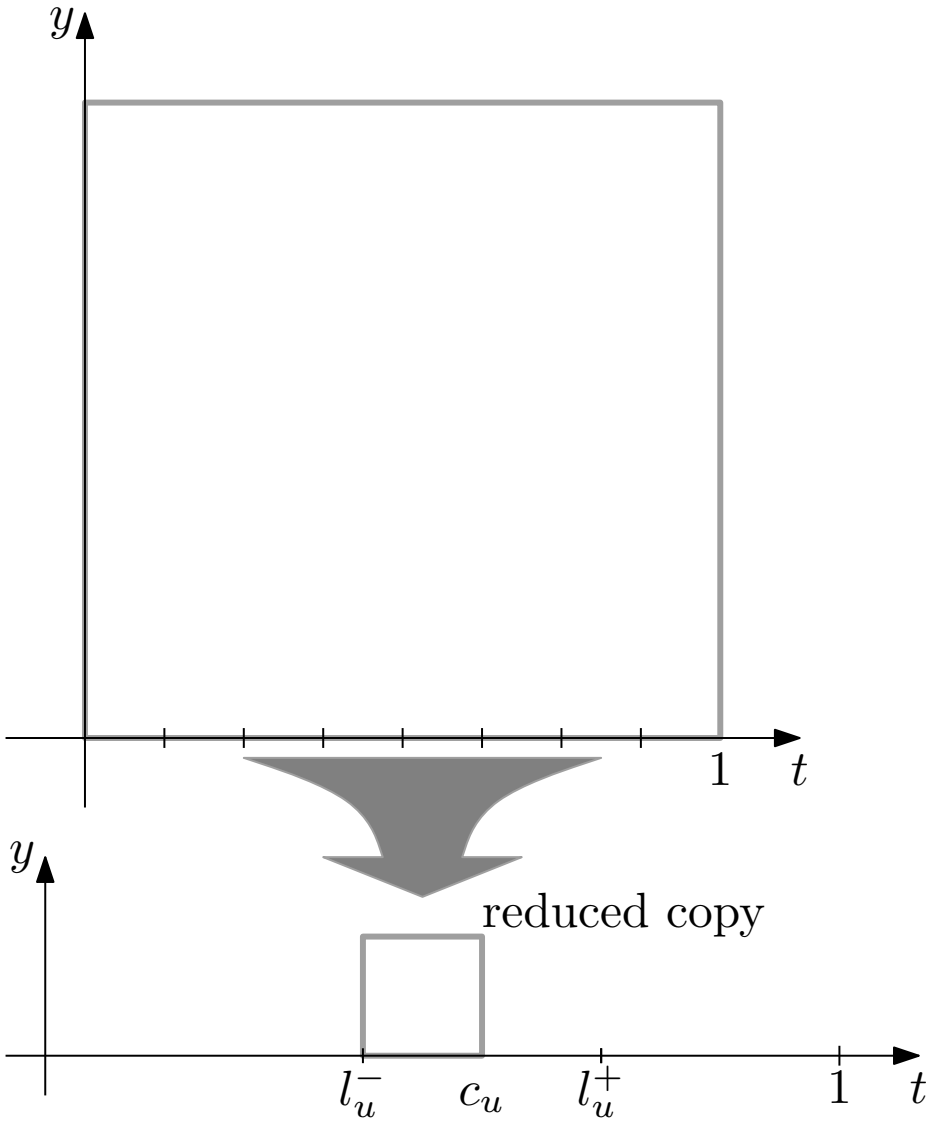

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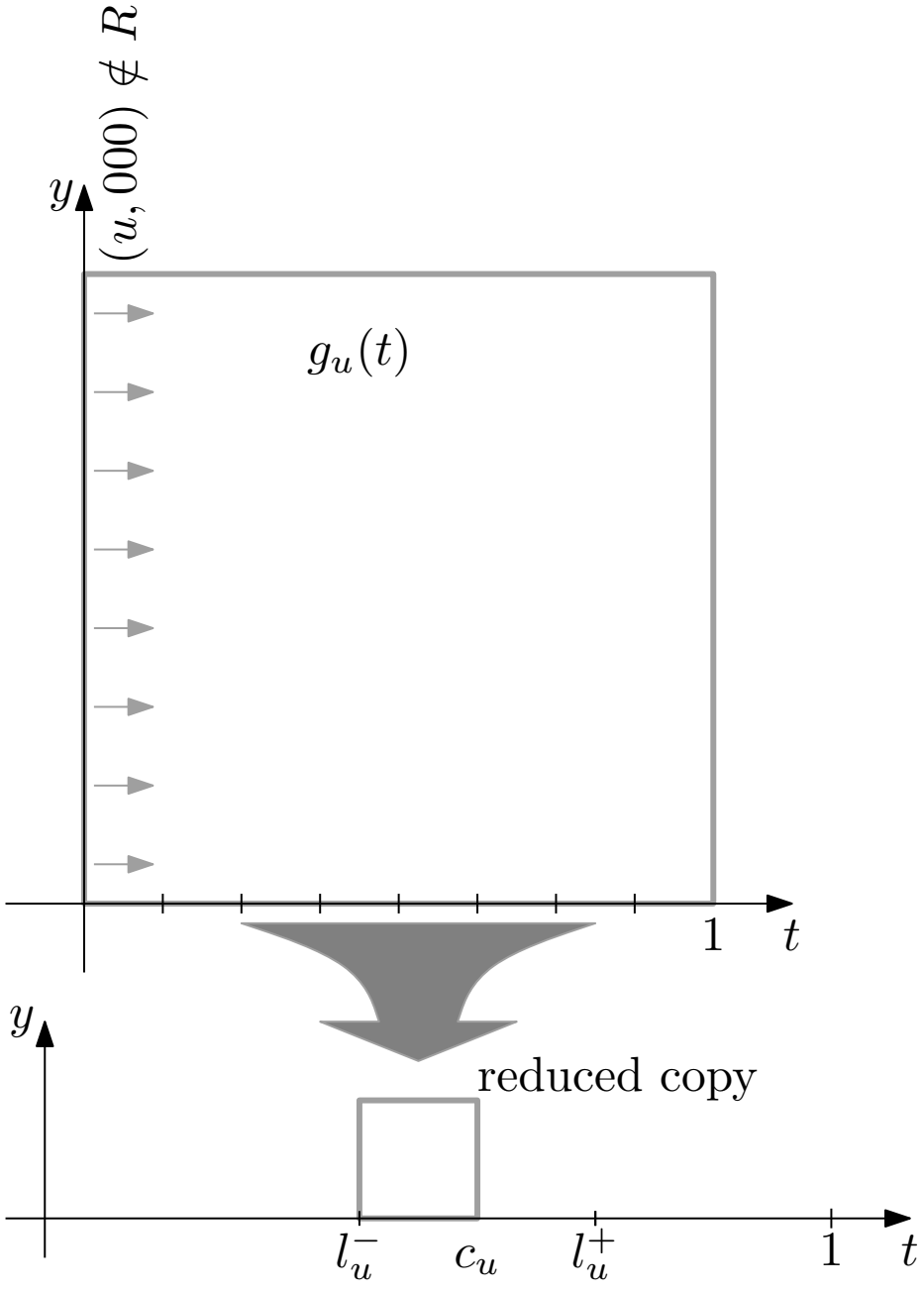

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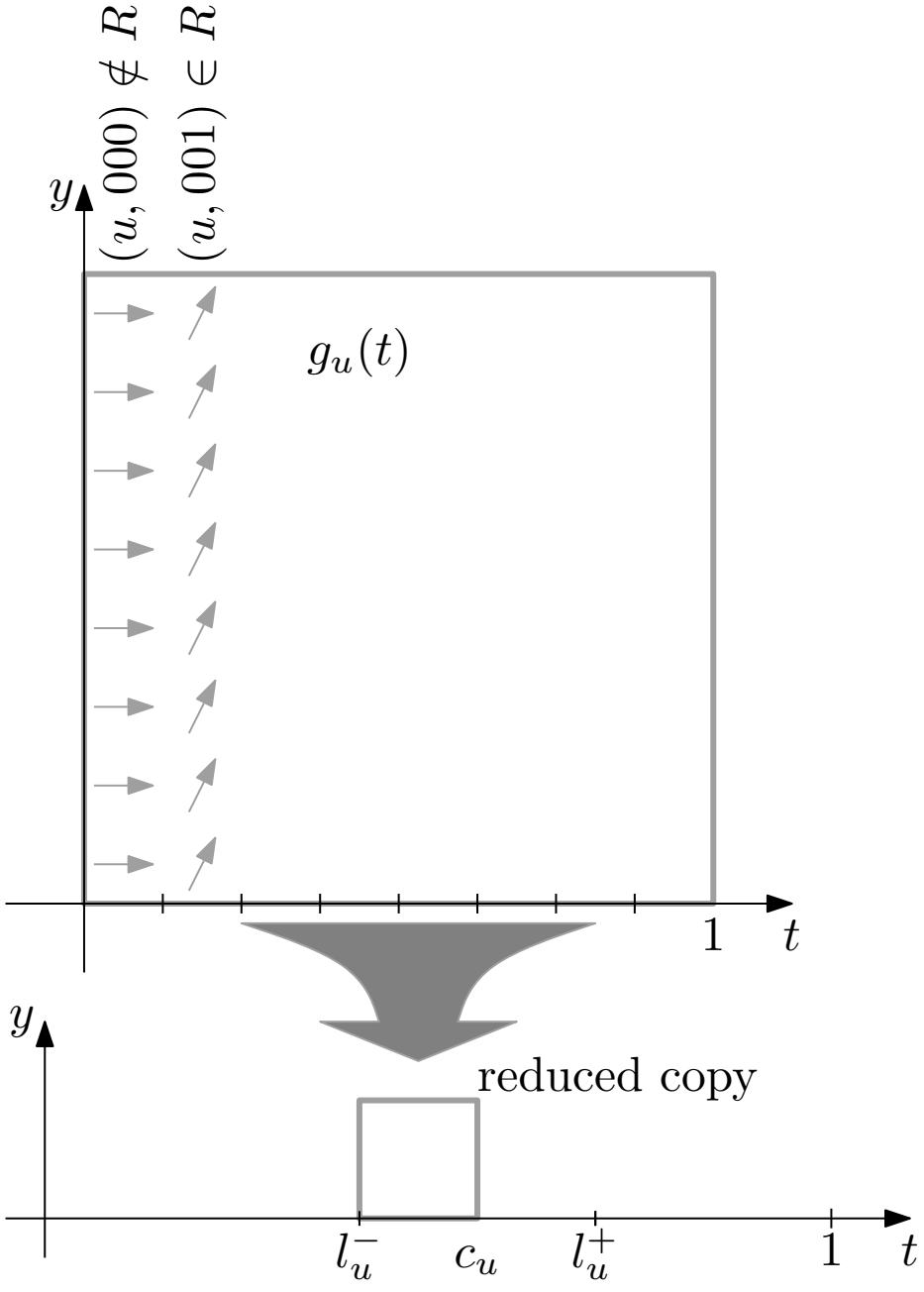

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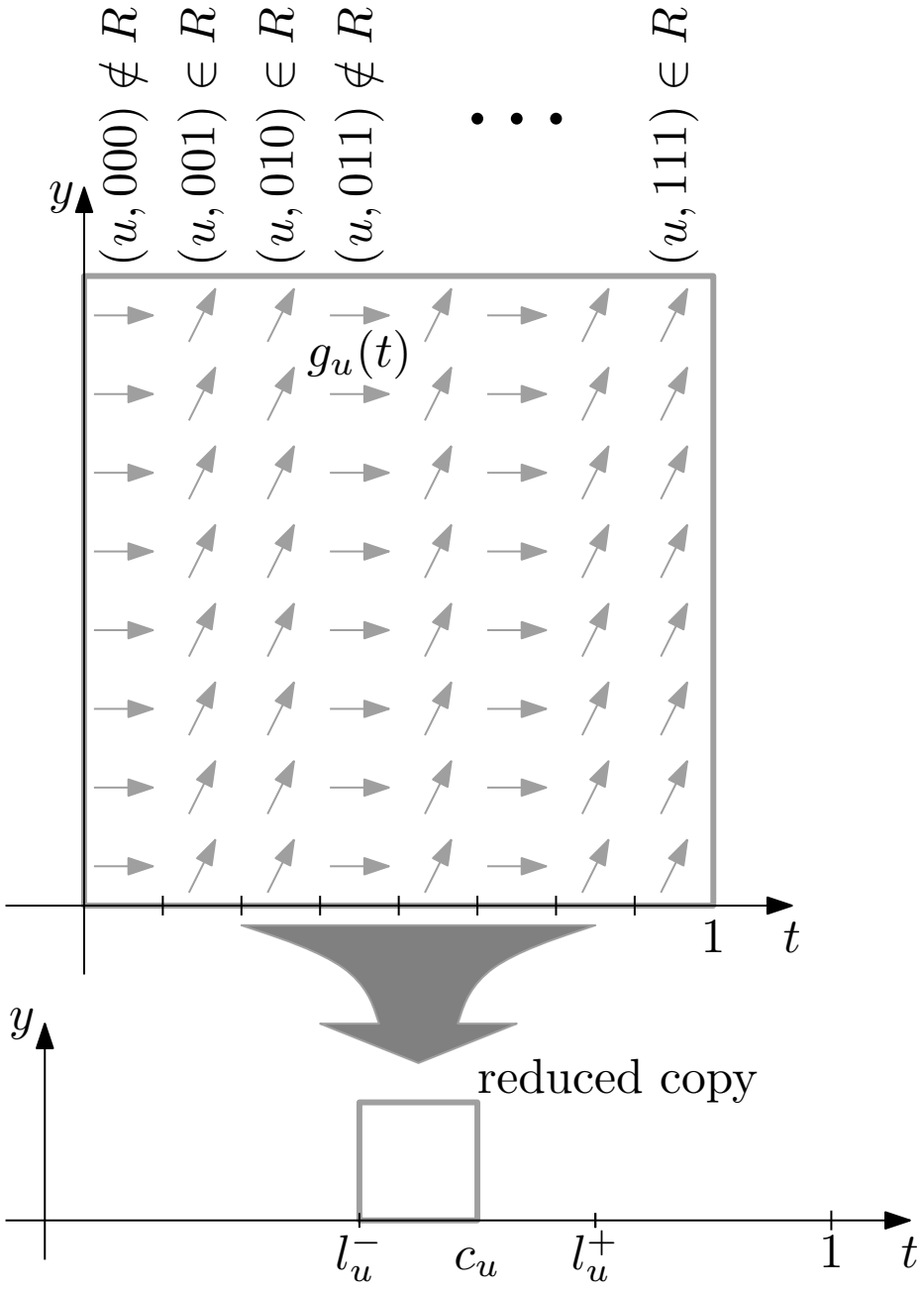

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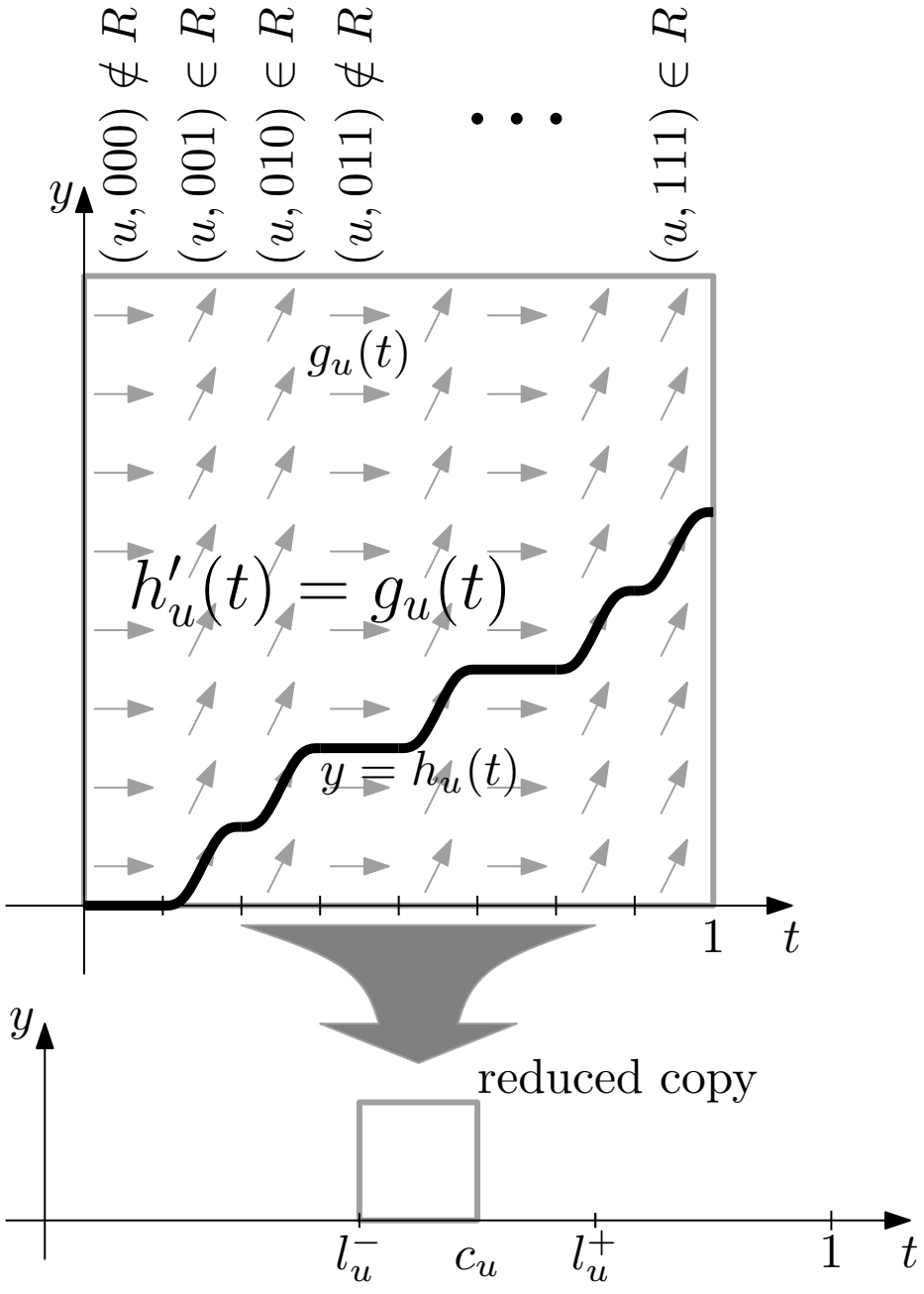

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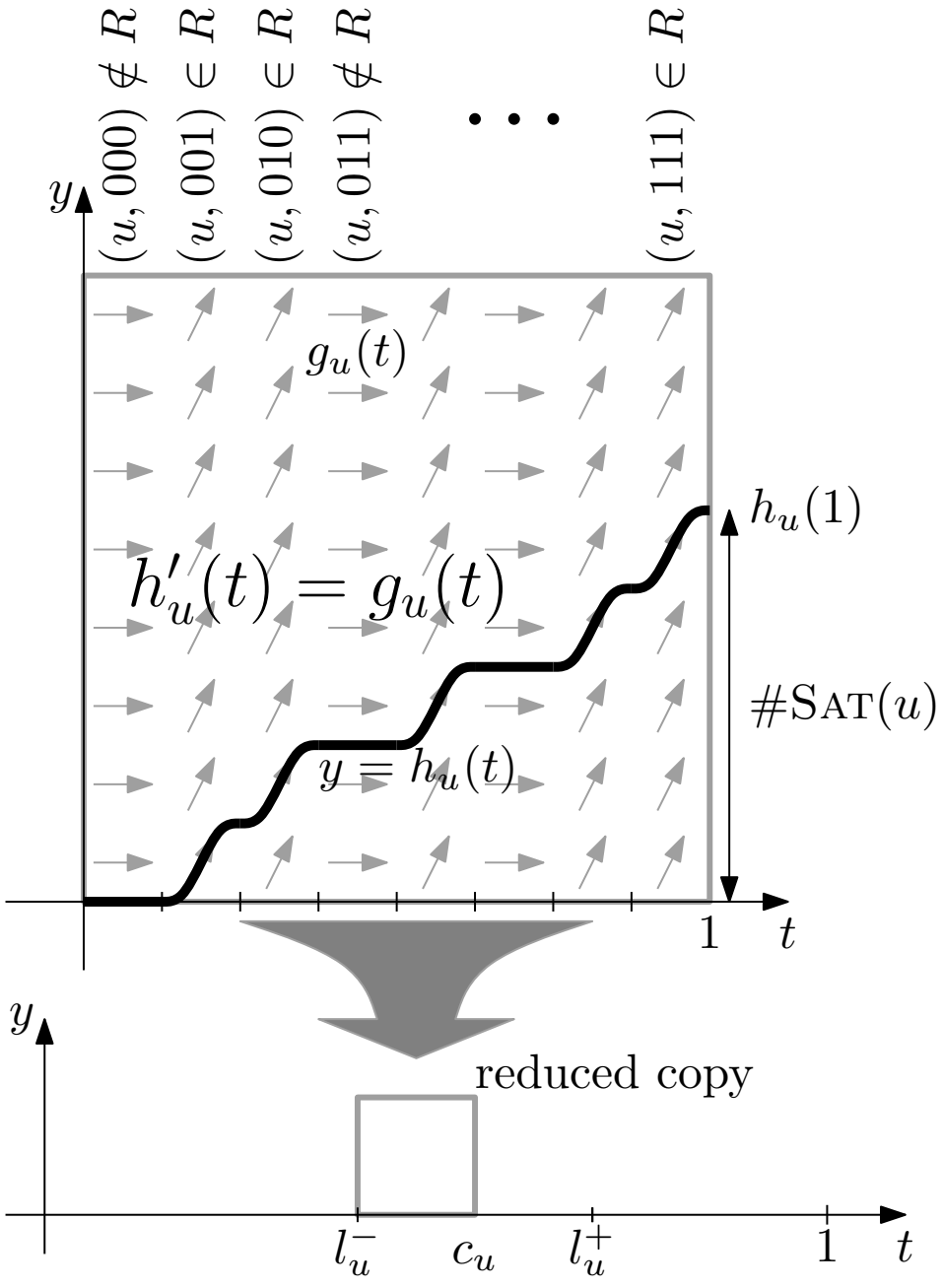

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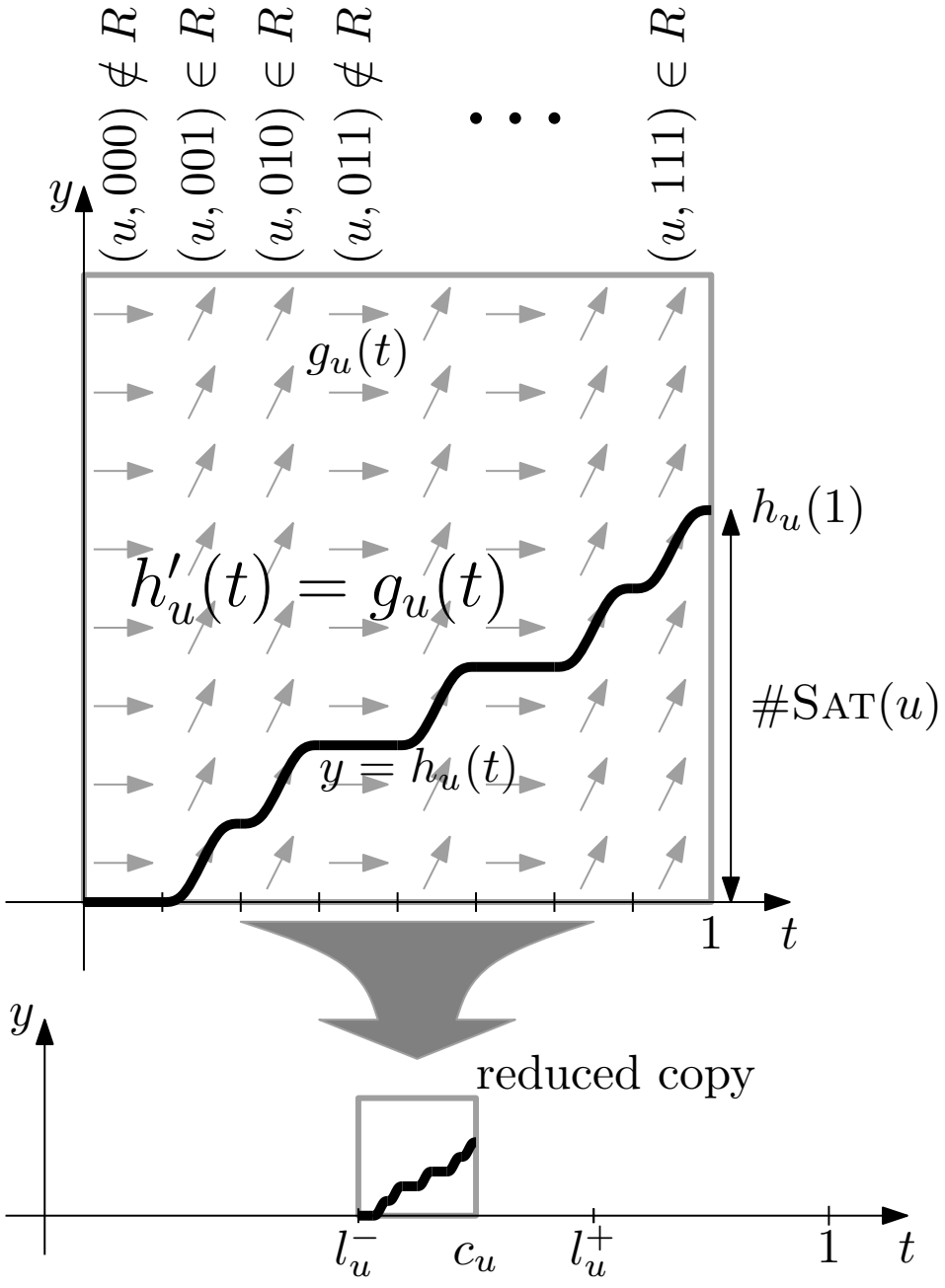

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$\#\textsc{Sat}(u)$ is encoded in $h_u(1)$, or in $h(c_u)$.

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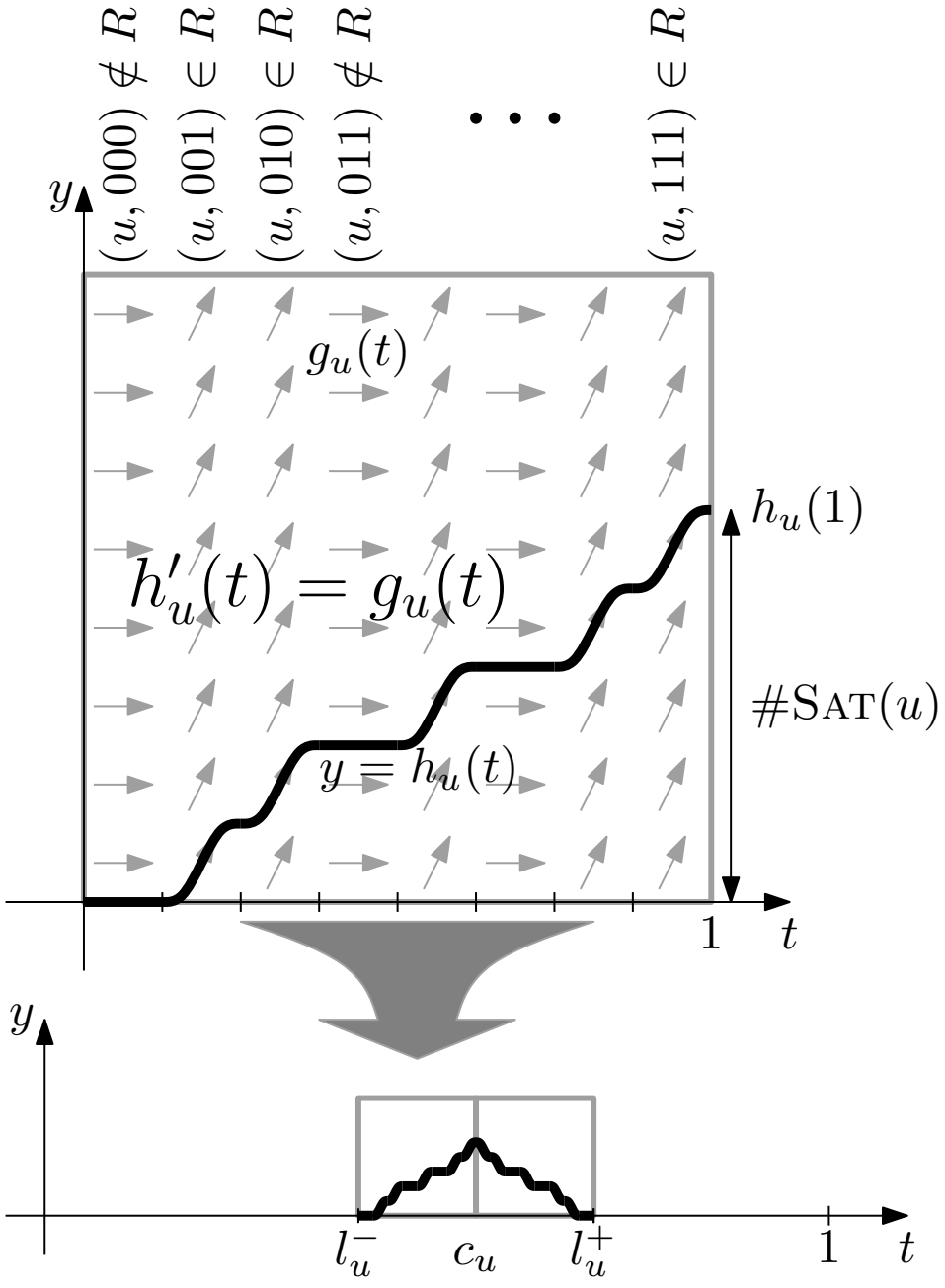

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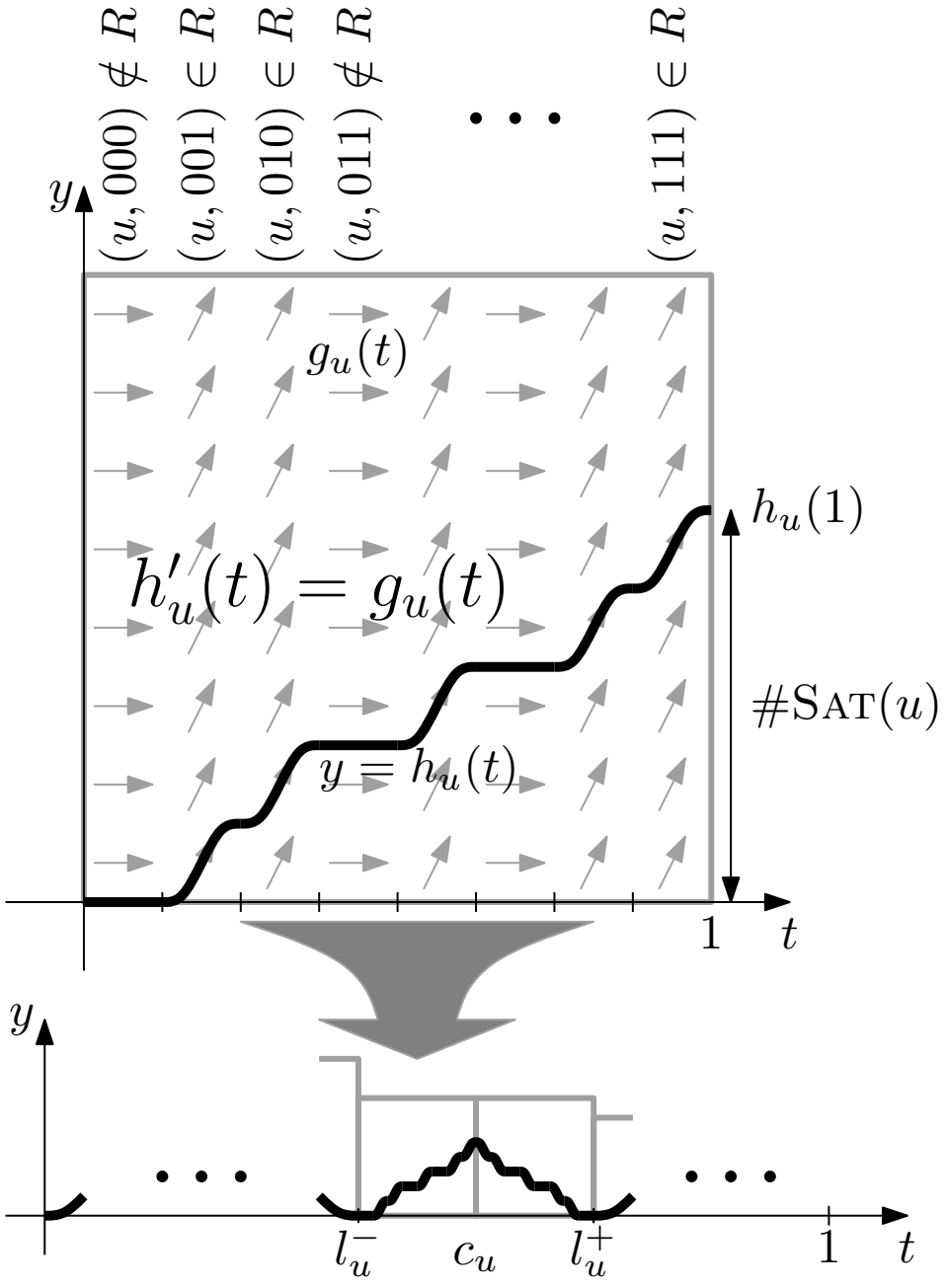

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Outline

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

1. Complexity of real functions
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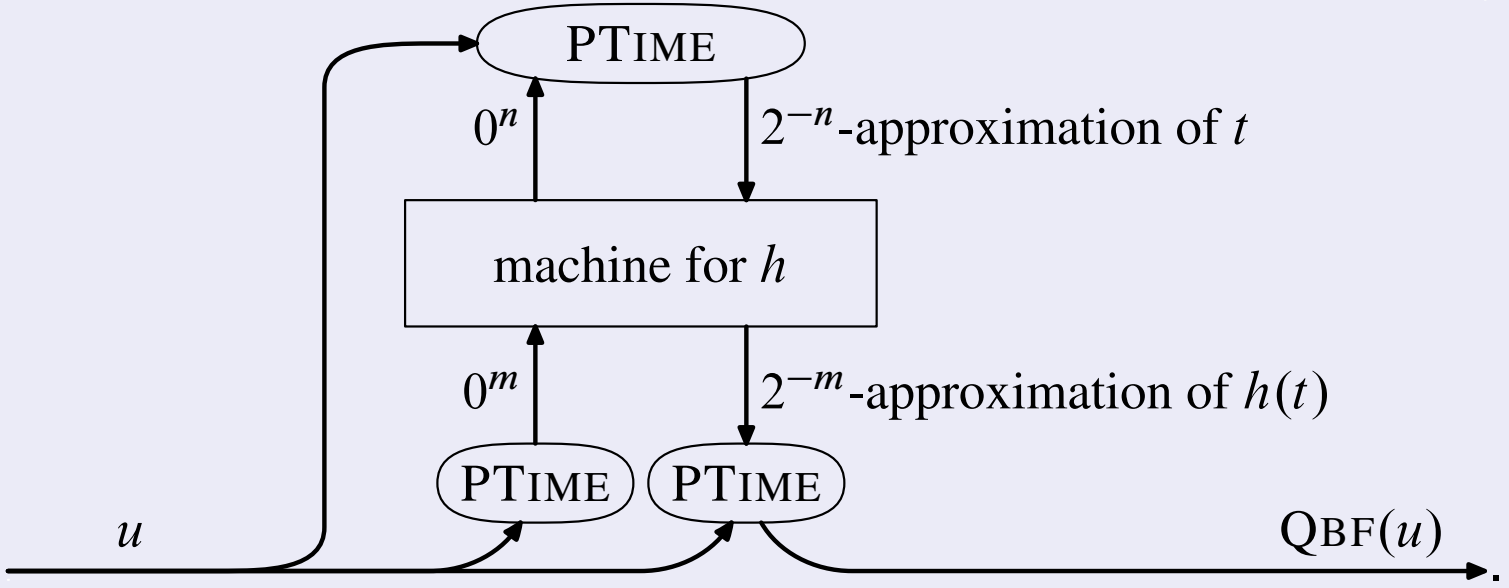
Complexity of Lipschitz IVPs

Main Theorem

There are $g\colon [0,1]\times[-1,1]\to \mathbf{R}$ and $h\colon [0,1]\to[-1,1]$ such that

- g is PTIME and Lipschitz continuous;
- $h(0)=0, \quad h'(t)=g\big(t,h(t)\big)$;
- h is PSPACE-hard in the sense that



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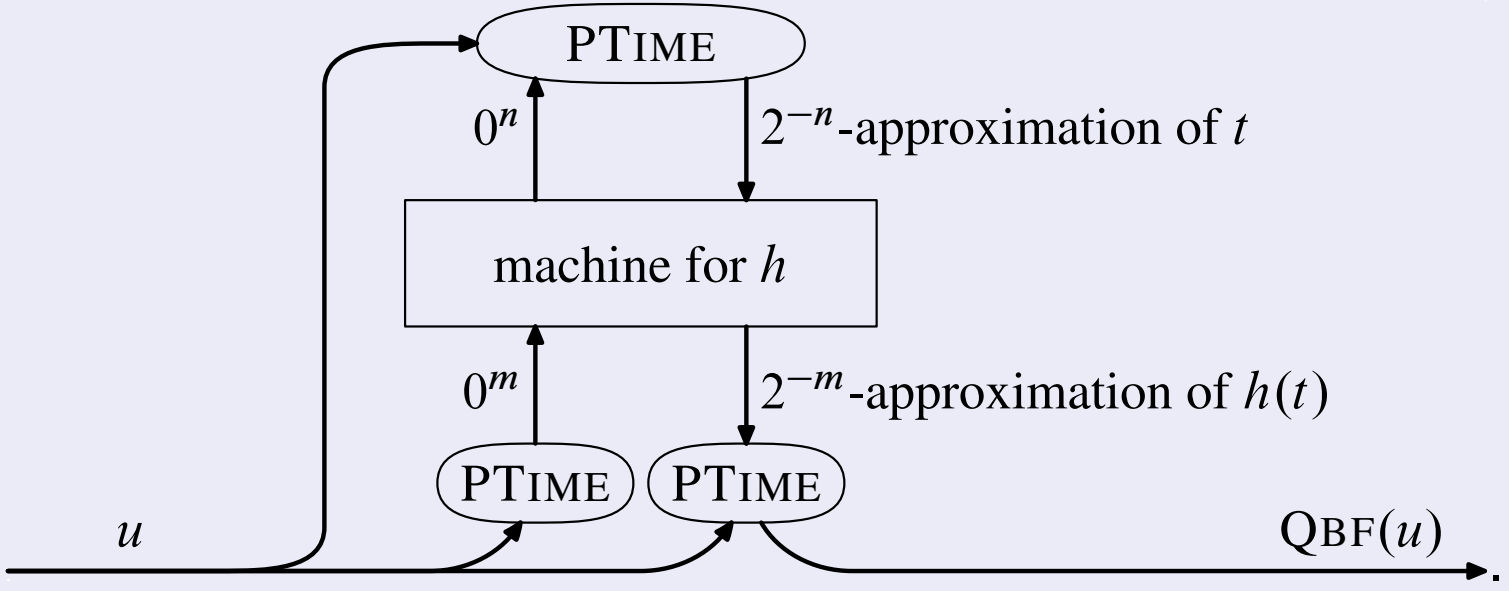
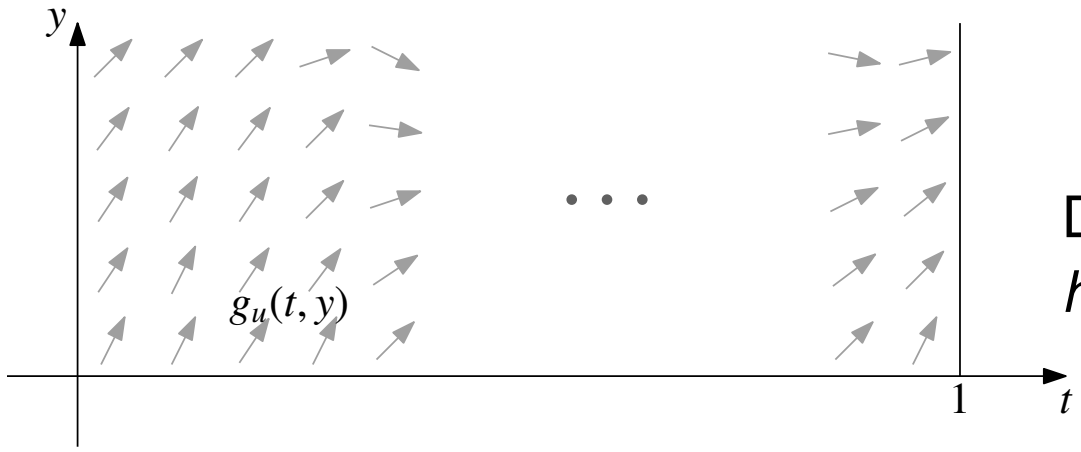


Cf. Upper bound: h is in PSPACE by the Euler method [Ko 1983].

Proof (1/3): An attempt to reduce PSPACE to IVP

As before, we need blocks g_u such that $h_u(1)$ indicates if $u \in$ QBF.



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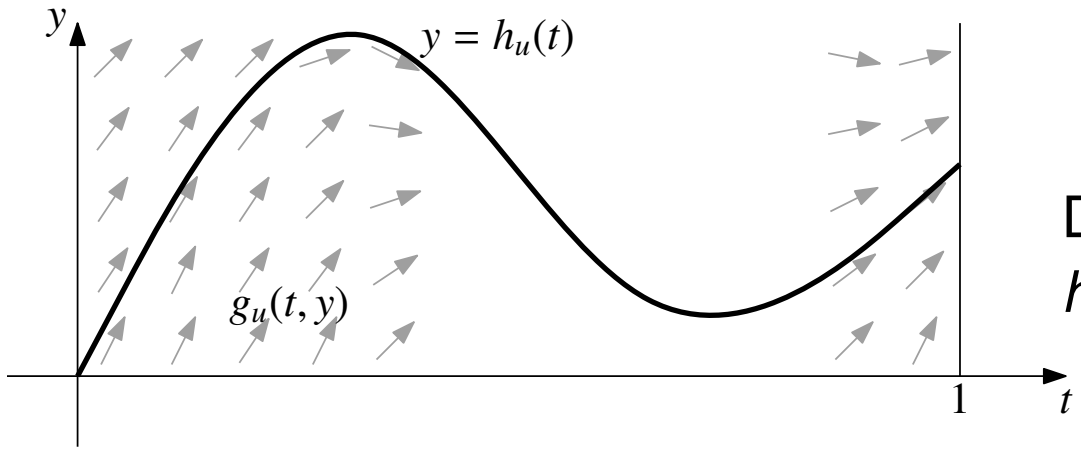

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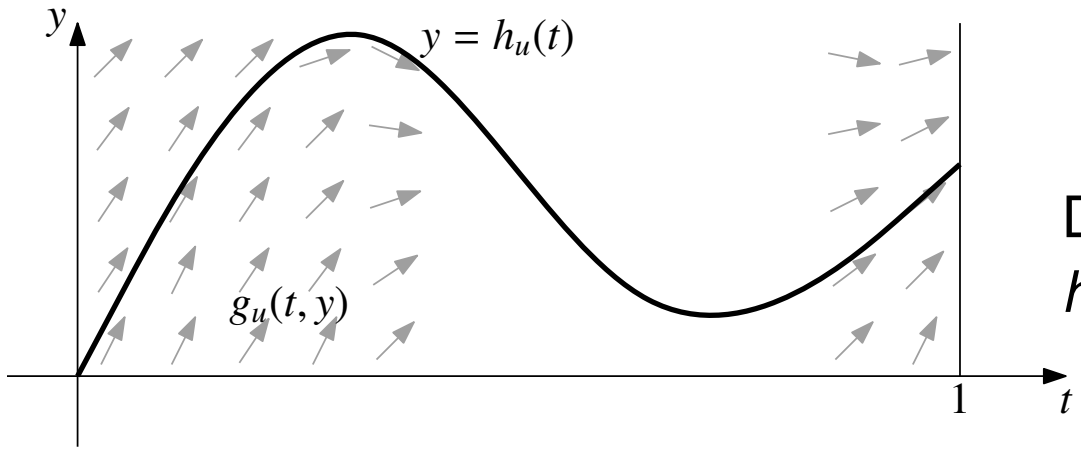

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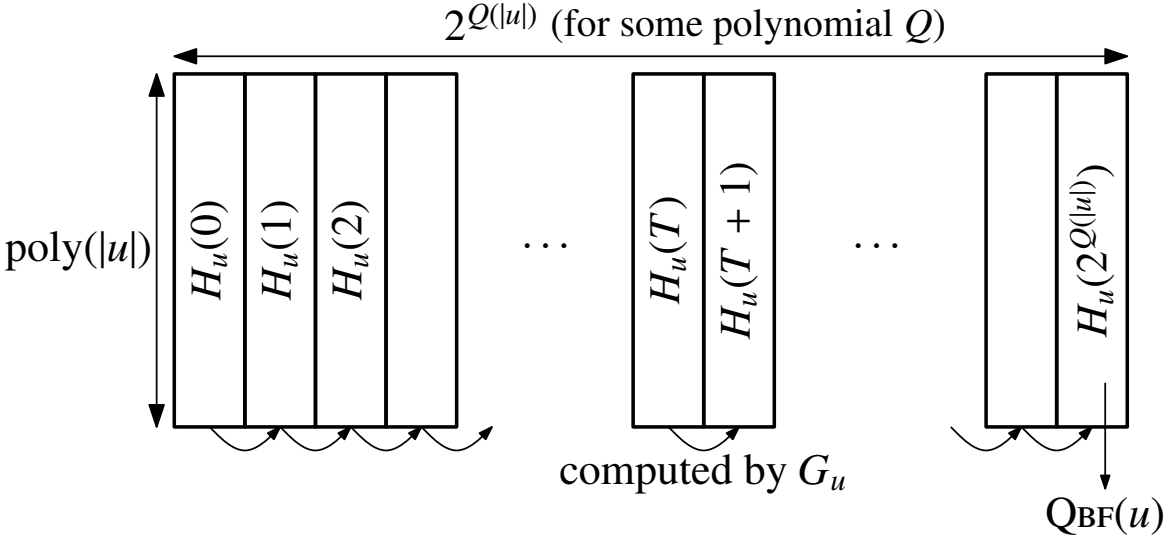

Description of a PSPACE machine on input u:
$H_u(T+1) = G_u(T, H_u(T))$

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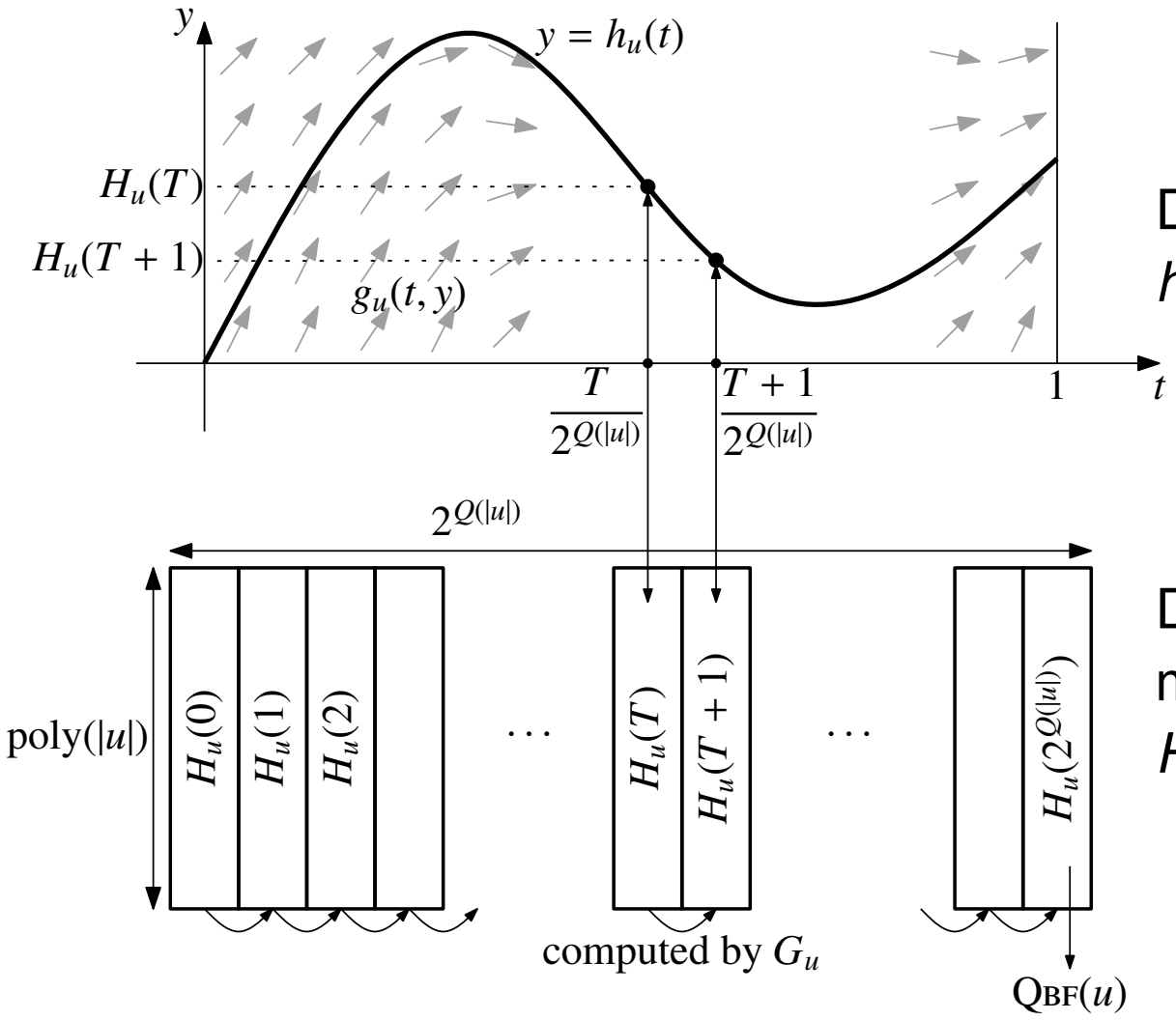

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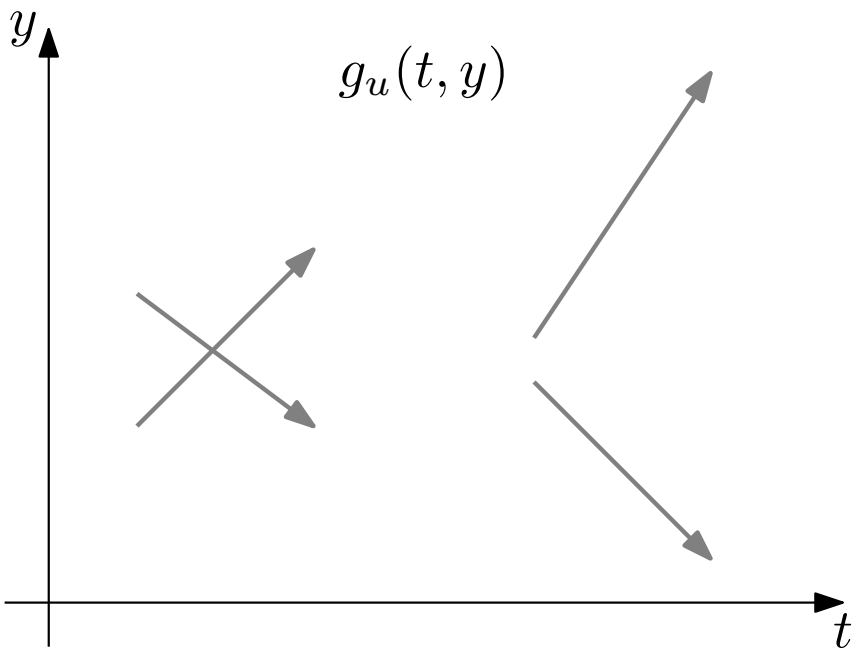
Proof (2/3): Why this attempt does not work

Not all G_u can be translated to g_u.



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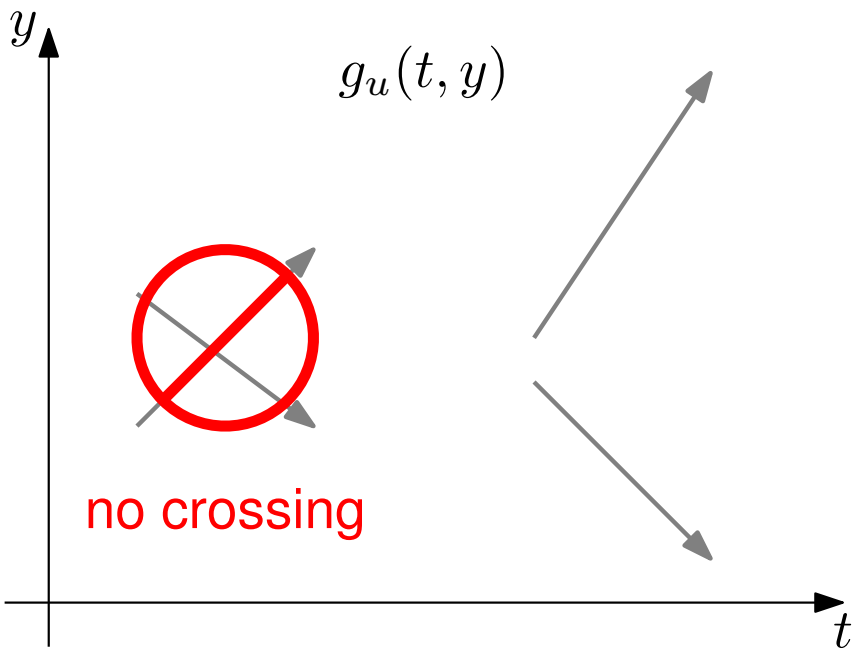

Flows can never cross.

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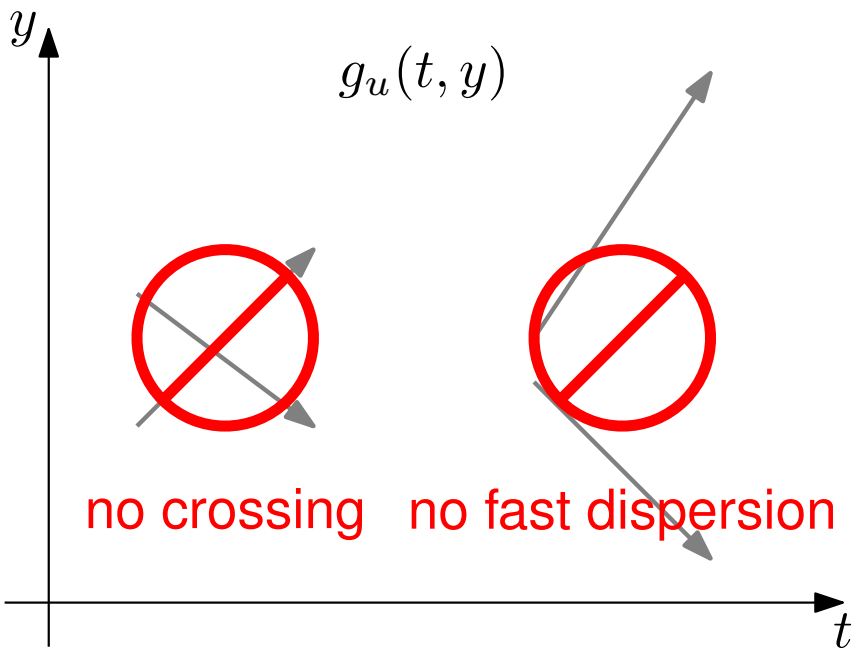

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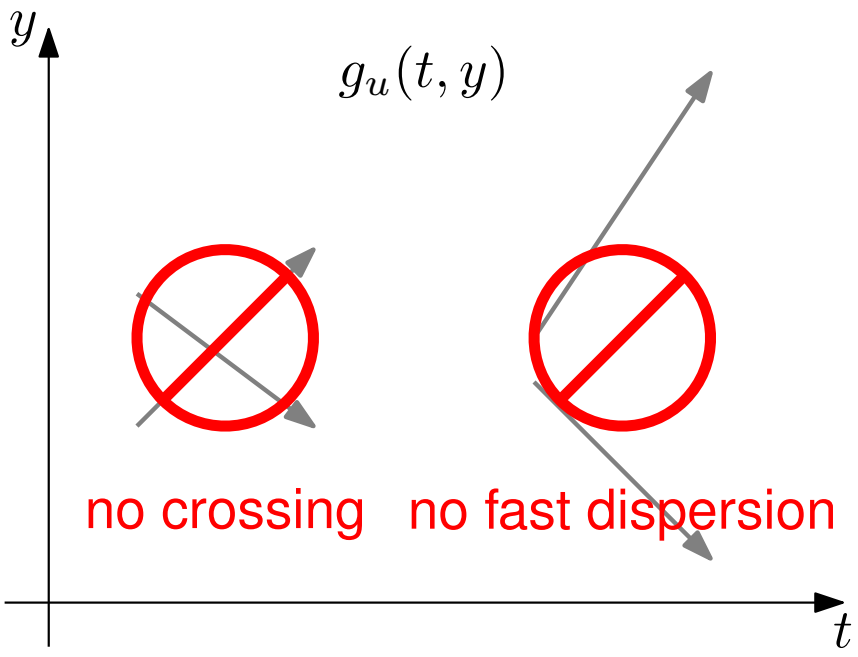

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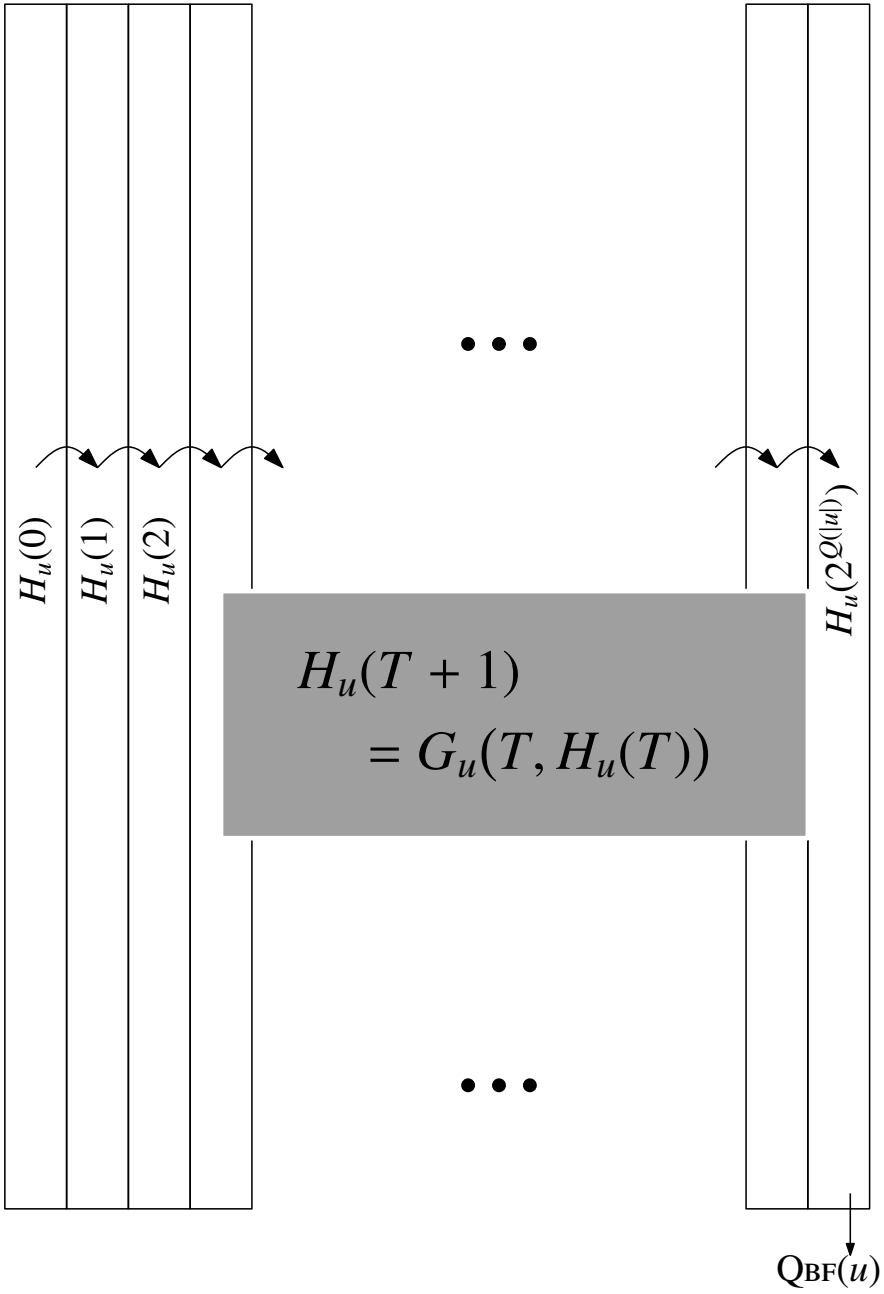
In fact, the Lipschitz condition keeps them from widening or narrowing fast.

The 'feedback' (on h_u) of equation $h'_u(t) = g_u\big(t, h_u(t)\big)$ is very weak under the Lipschitz condition.

Proof (3/3): Layered PSPACE tableaux



General PSPACE computation tableaux cannot be simulated by the differential equation.

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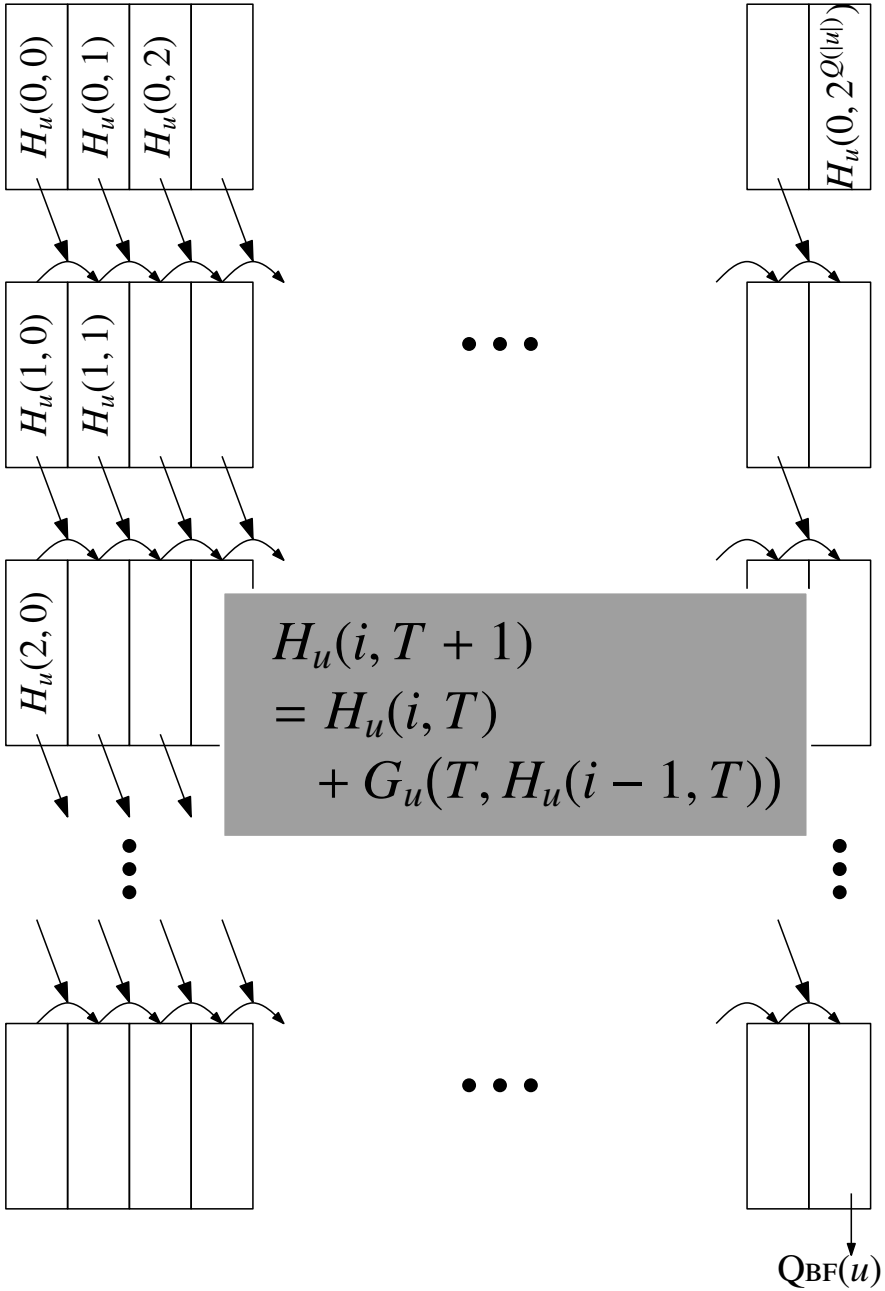

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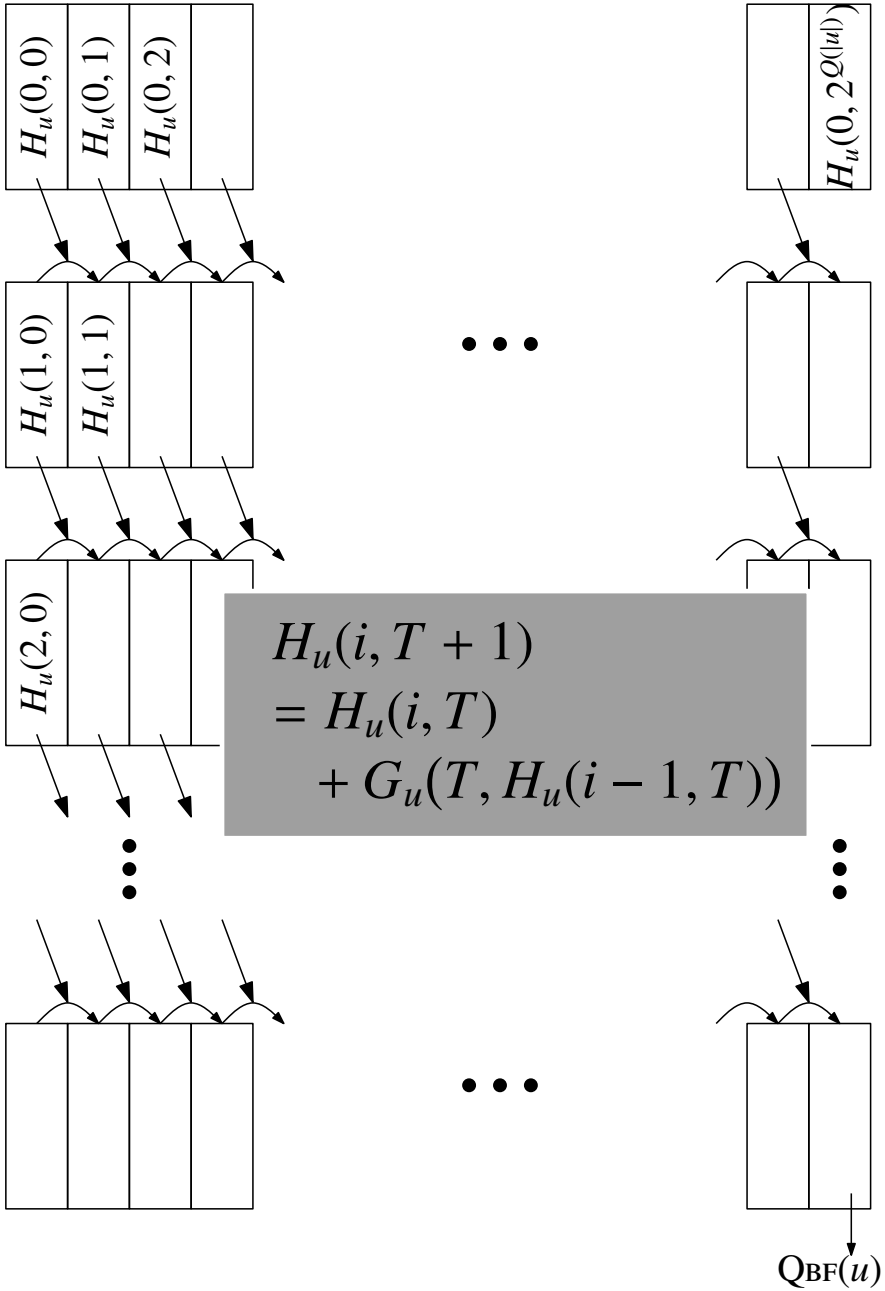

General PSPACE computation tableaux cannot be simulated by the differential equation.

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This tableau can be simulated by the equation, and is PSPACE-complete despite the restriction.

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Related results

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

Assuming g is PTIME, how complex can h be?

Assumptions	Positive results on h	Negative results on h
None	——	can be (non-unique and) all non-computable [Pour-El 1979]
h is the unique solution	computable [implicit in Osgood 1898]	time (or space) cannot be bounded [Miller 1970]
g is Lipschitz continuous	PSPACE [Ko 1983]	can be PSPACE-hard
g is analytic	PTIME [Ko 1988, K.]	——

Open problems

- What happens between the Lipschitz case (PSPACE) and the analytic case (PTIME)?
- Effective versions? (Complexity of "computing h from g")
- Other differential equations