Distance k-sectors exist

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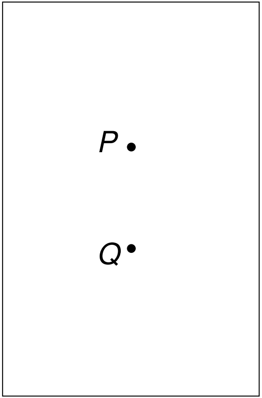
Distance *k*-sectors [AMT06]



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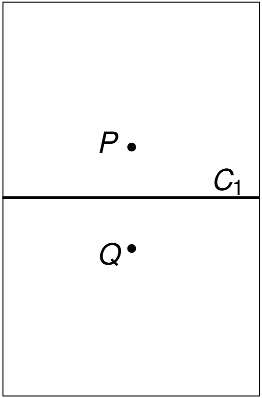

Bisector:

$$C_1 = \text{bisect}(P, Q)$$
$$:= \{ x \in \mathbb{R}^2 : \text{dist}(x, P) = \text{dist}(x, Q) \}$$

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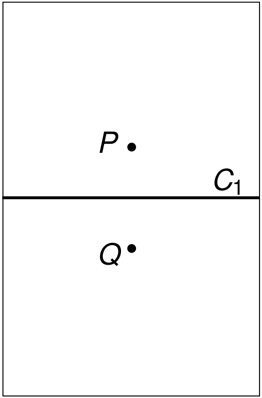

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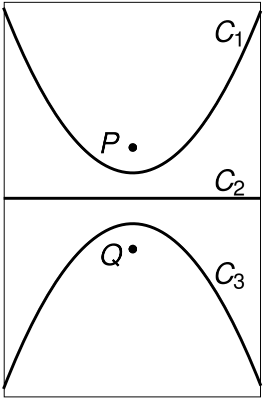

4-sector:

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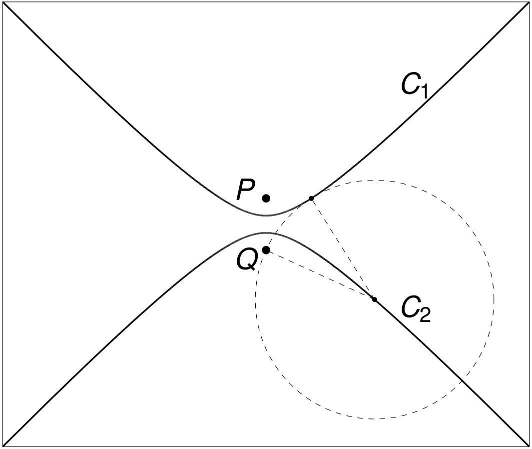
$C_3 = \text{bisect}(C_2, Q)$

Existence is not trivial



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$C_1 = \text{bisect}(P, C_2)$

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Exists and is unique [AMT06]

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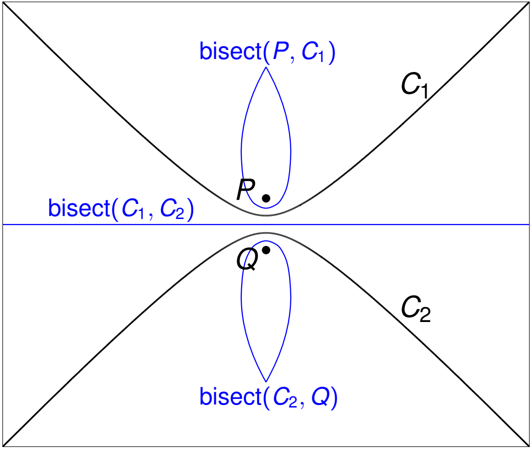

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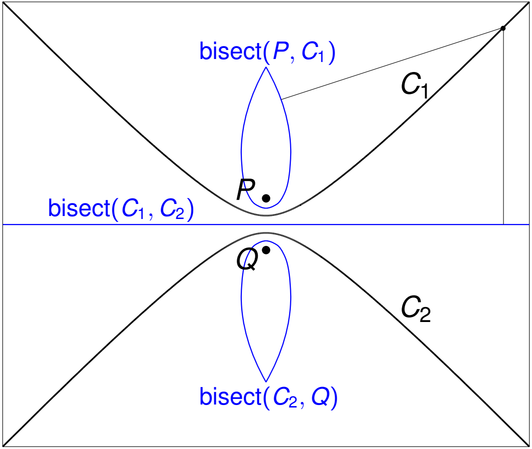

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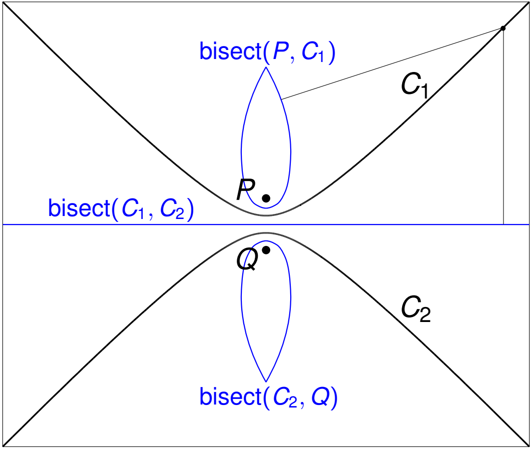

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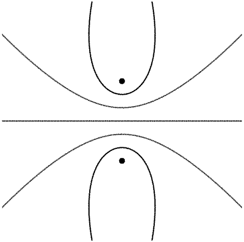

True 6-sector
[COT07]

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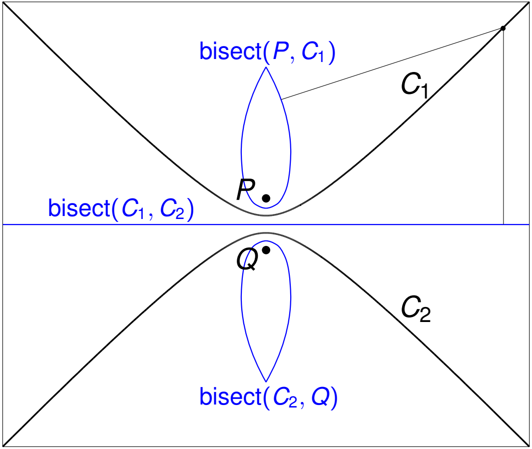

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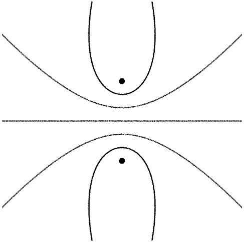


True 6-sector
[COT07]

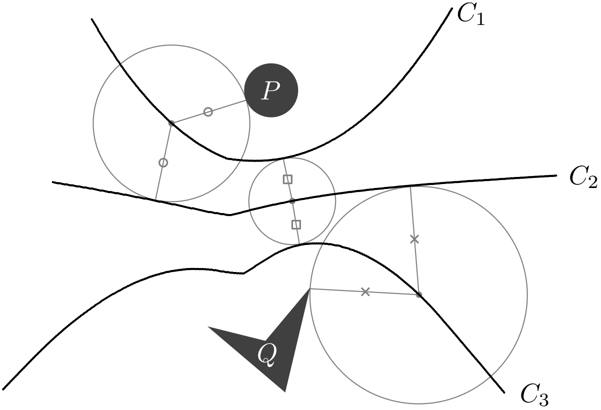
Question

Do k-sectors exist?

Our result

Main Theorem (existence of *k*-sectors)

For any nonempty disjoint closed sets P, $Q \subseteq \mathbb{R}^2$,
there is a k-sector.



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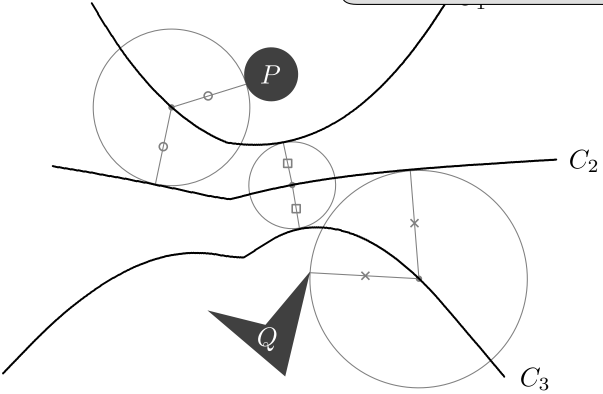
For any nonempty disjoint closed sets P, $Q \subseteq \mathbb{R}^2$, there is a k-sector.

> can be generalized to $\mathbb{R}^d$ and proper geodesic spaces



Proof: k-sector as a Tarski fixed point

A k-sector is a fixed point of the function **Bisect** given by

$$\mathbf{Bisect}(C_1, \ldots, C_{k-1}) = \big(\,\mathrm{bisect}(P, C_2), \\ \mathrm{bisect}(C_1, C_3), \\ \ldots, \\ \mathrm{bisect}(C_{k-2}, Q)\big).$$

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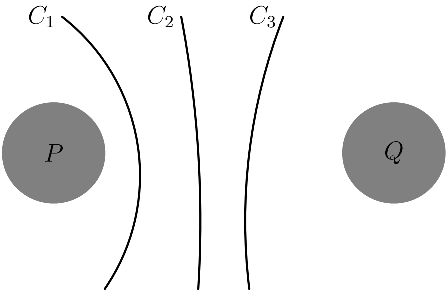

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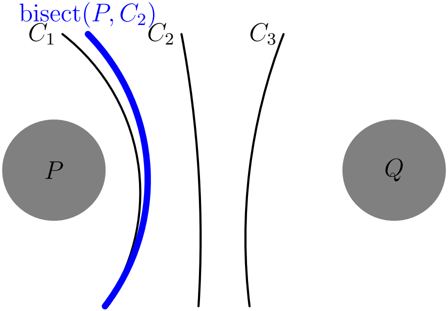

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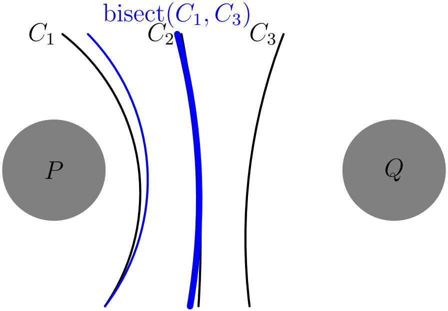

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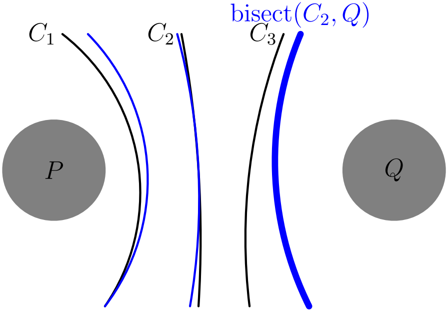

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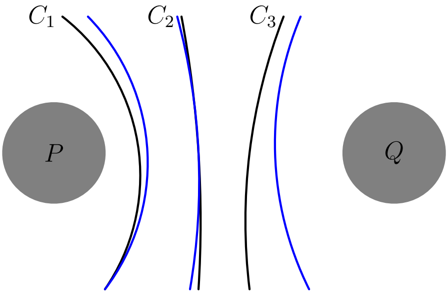

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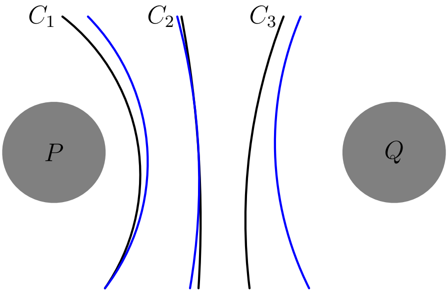

Hence, it exists by the Tarski fixed point theorem (next slide). □

Applying Tarski's theorem

Tarski fixed point theorem [Tar55]

A monotone function on a complete lattice has a fixed point.

- $(L, \sqsubseteq)$ *complete lattice*: Every $X \subseteq L$ has sup and inf
- $f\colon L \to L$ *monotone*: $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$

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Here, let L be the set of $(k-1)$-tuples of curves $(C_1, \ldots, C_{k-1})$ and define the order by

$$(C_1, \ldots, C_{k-1}) \sqsubseteq (C'_1, \ldots, C'_{k-1}) \iff \text{Each } C_i \text{ lies to the left of } C'_i \,.$$

Then **Bisect** is monotone.

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Here, let L be the set of $(k-1)$-tuples of ~~curves $(C_1, \ldots, C_{k-1})$~~ sets $(R_1, \ldots, R_{k-1})$ and define the order by

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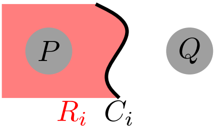
To be precise, we really work on the region R_i to the left of C_i.



Drawing *k*-sectors

Tarski's theorem gives the fixed point

$$\bigsqcup \{\, x \in L : x \sqsubseteq \mathbf{Bisect}(x) \,\}.$$

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But how can we draw a k-sector?

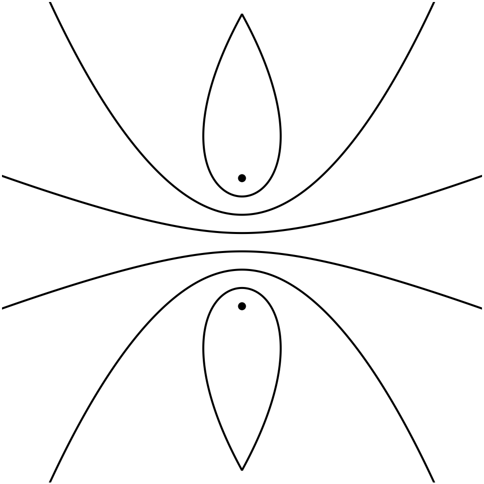
Proposition 5 (obtaining a k-sector by iteration)

Let $\perp = (P, \ldots, P)$. The sequence

$$\perp \quad \sqsubseteq \quad \mathbf{Bisect}(\perp) \quad \sqsubseteq \quad \mathbf{Bisect}(\mathbf{Bisect}(\perp)) \quad \sqsubseteq \quad \cdots$$

converges to a fixed point of **Bisect** (i.e., a k-sector).

Directions for further work



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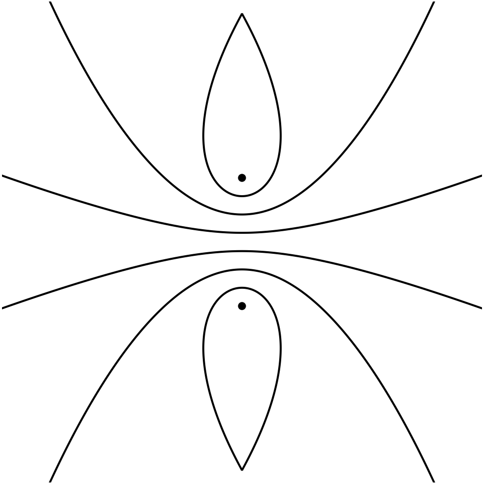

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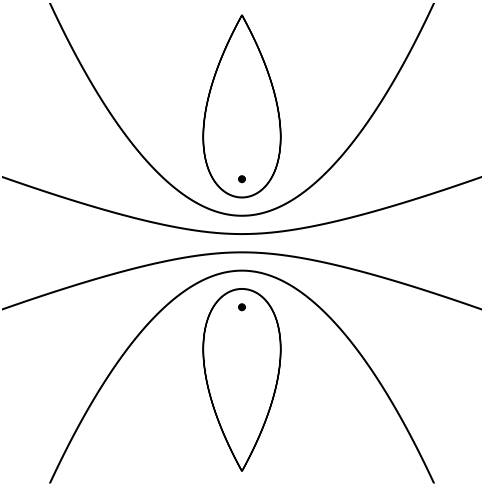

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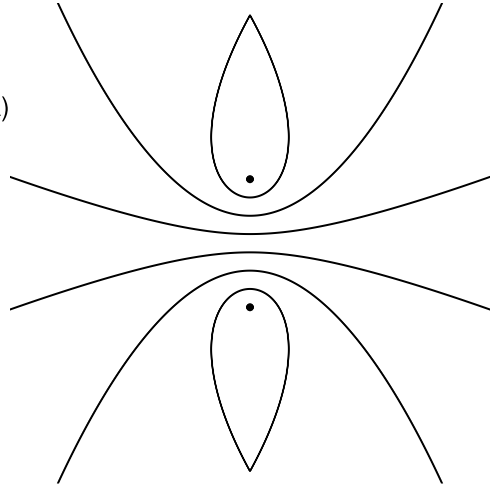

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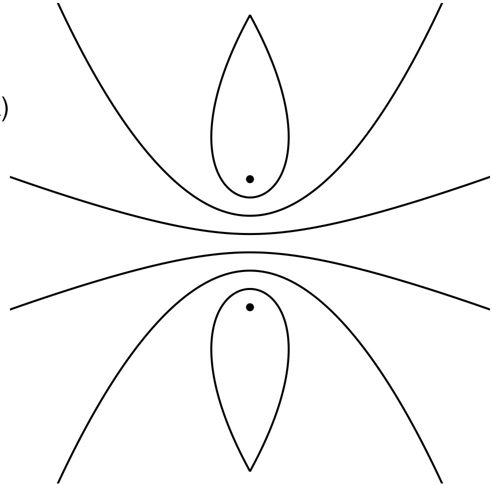

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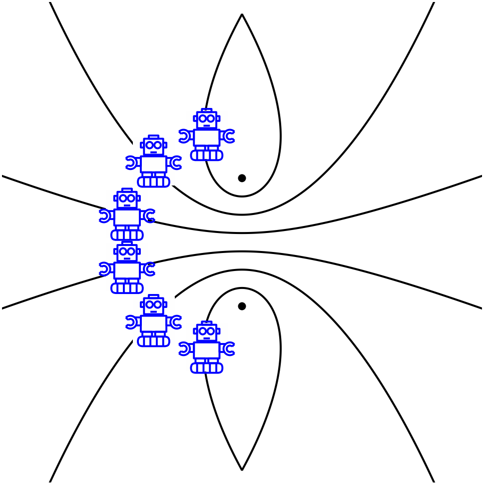

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