Distance *k*-Sectors and Zone Diagrams

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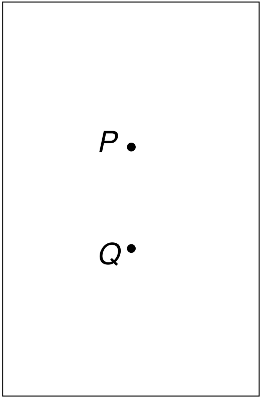
Distance k-Sectors

Distance *k*-sectors [Asano et al., STOC 2006]



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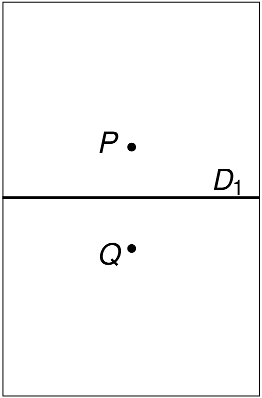
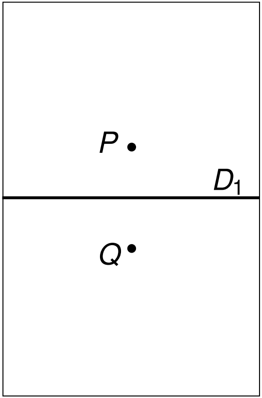


bisector (2-sector):

$$D_1 = \mathrm{dom}(P, Q)$$
$$:= \{\, x \in \mathbb{R}^d : \mathrm{dist}(x, P) \leq \mathrm{dist}(x, Q) \,\}$$

Distance k-sectors [Asano et al., STOC 2006]



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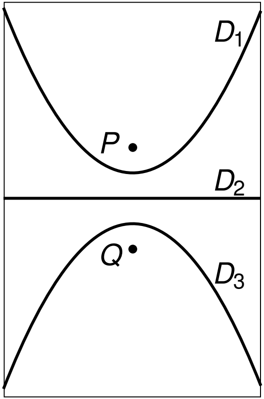
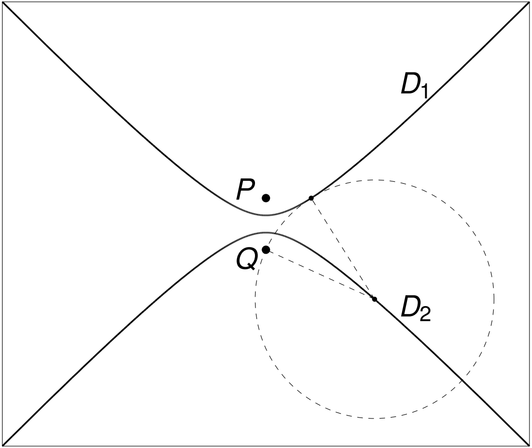


4-sector:

$$D_1 = \mathrm{dom}(P, D_2^c)$$
$$D_2 = \mathrm{dom}(D_1, D_3^c)$$
$$D_3 = \mathrm{dom}(D_2, Q)$$

Existence of k-sectors is not trivial



trisector (3-sector):

$D_1 = \mathrm{dom}(P, D_2^c)$

$D_2 = \mathrm{dom}(D_1, Q)$

Existence and uniqueness [ibid.]

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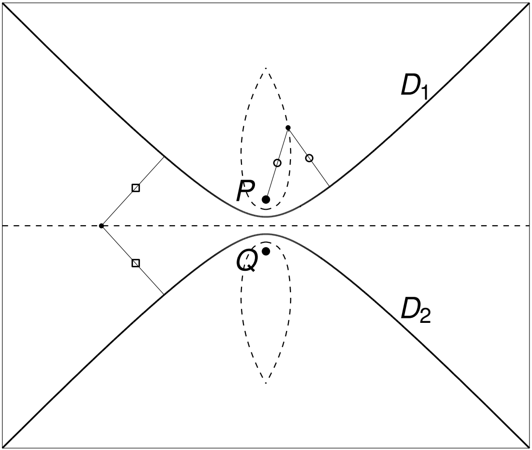

A 6-sector?

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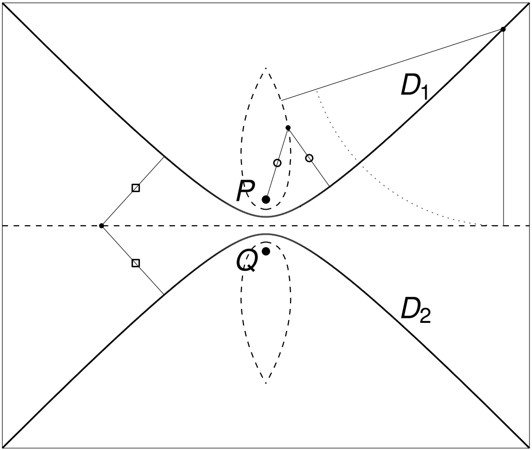

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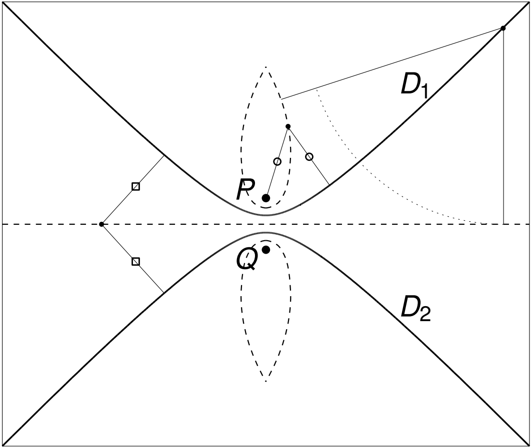

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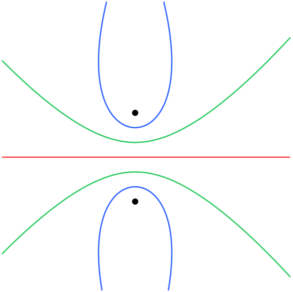

6-sector

[Chun et al., ISVD 2007]

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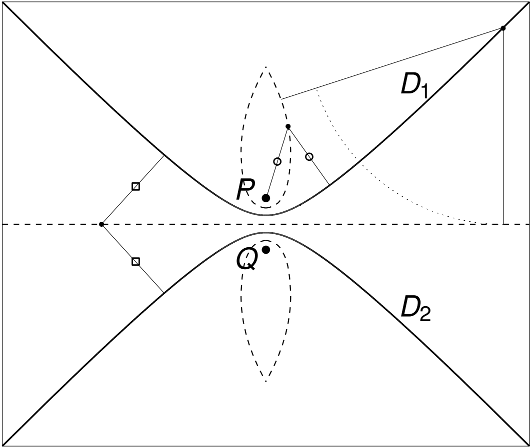

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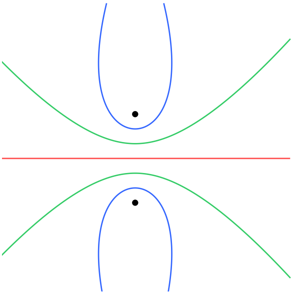


6-sector
[Chun et al., ISVD 2007]

Problem: Do they exist for all k?

Our result 1: existence of k-sectors

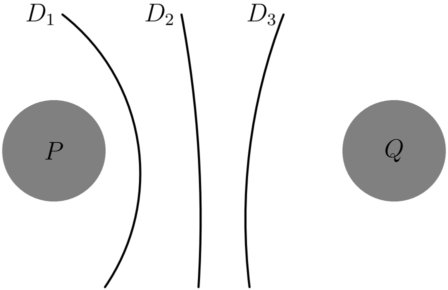
Theorem

For any positively separated sets P, $Q \subseteq \mathbb{R}^d$,
there is a k-sector between them.

Proof idea: k-sector as a Tarski fixed point

A k-sector is a fixed point of the function **Dom** defined by

$$\textbf{Dom}(D_1, \ldots, D_{k-1}) = (E_1, \ldots, E_{k-1}),$$
$$\text{where } E_i := \text{dom}(D_{i-1}, D^c_{i+1}).$$



So we can apply:

Tarski's Fixed Point Theorem (1955)

A monotone function on a complete lattice has a fixpoint.

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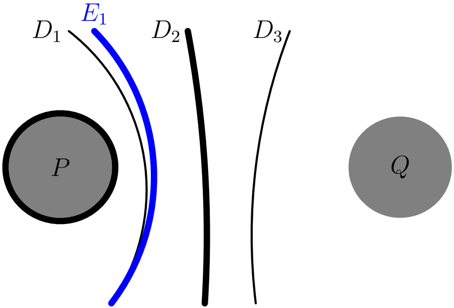

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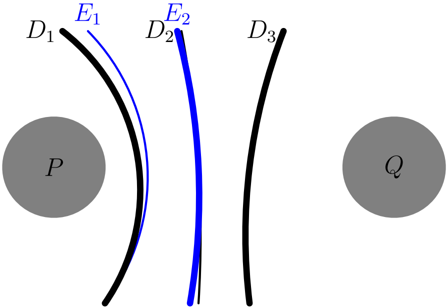

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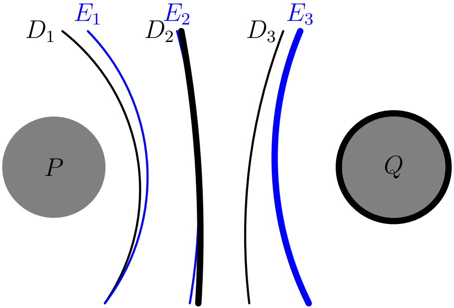

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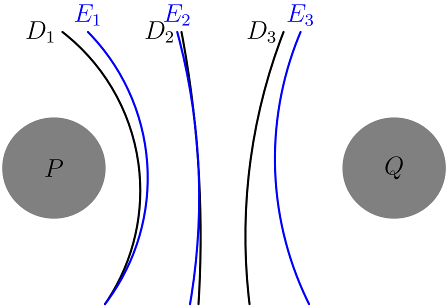

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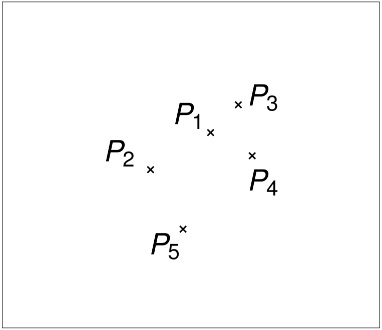
Zone Diagrams

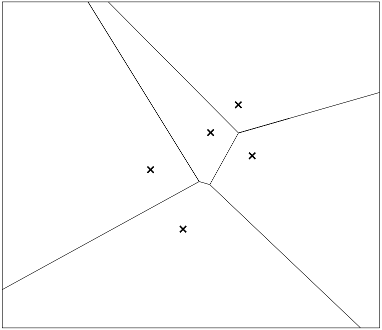
Zone diagrams [Asano et al., SODA 2007]



Voronoi diagram

$$V_i = \text{dom}\left(P_i, \bigcup_{j \neq i} P_j\right)$$

Zone diagrams [Asano et al., SODA 2007]



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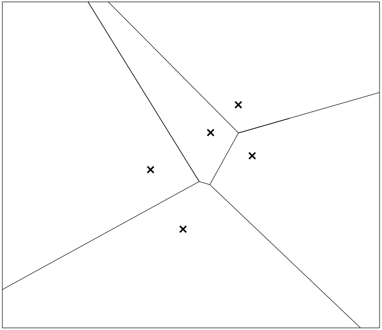

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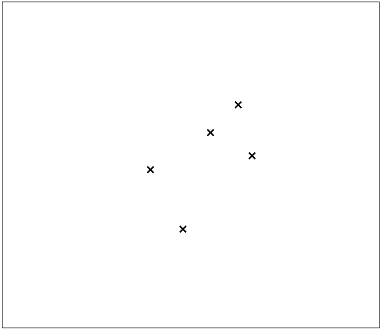
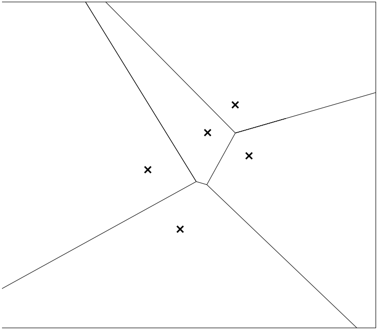


zone diagram

$$Z_i = \mathrm{dom}\Big(P_i, \bigcup_{j \neq i} Z_j\Big)$$

Zone diagrams [Asano et al., SODA 2007]



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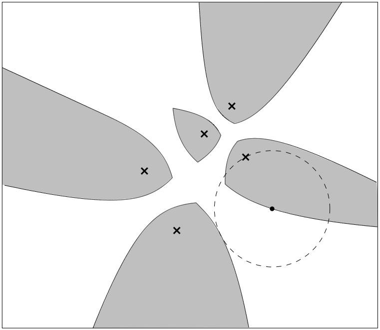

zone diagram

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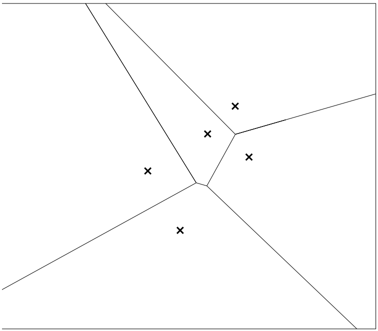
Zone diagrams [Asano et al., SODA 2007]



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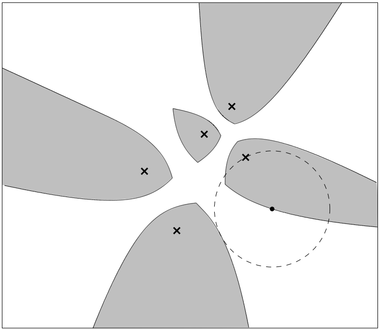

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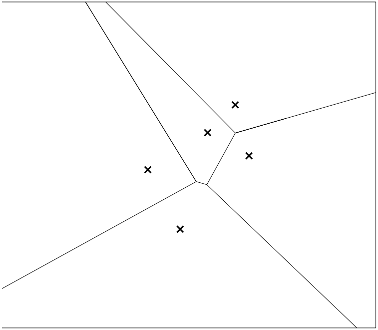
Zone diagram for two sites = distance trisector.

Zone diagrams [Asano et al., SODA 2007]



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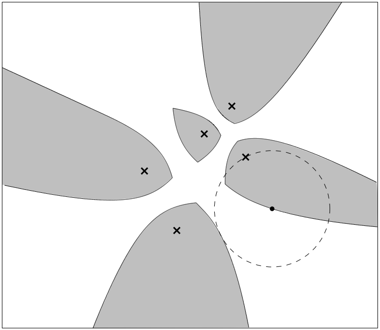

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$$Z_i = \mathrm{dom}\Big(P_i, \bigcup_{j \neq i} {\color{red} Z_j}\Big)$$

Problem: Does it exist?

(Some progress: [ibid.], [Chun et al. 2007], [Reem and Reich 2009])

Our result 2: existence and uniqueness of the ZD

Theorem

For any positively separated sets $P_1, \ldots, P_n \subseteq \mathbb{R}^d$,
there is a zone diagram of them and it is unique.

Proof: Tarski theorem + geometric argument

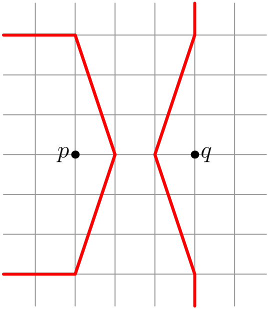
A zone diagram is a fixed point of **Dom** defined by

$$\mathbf{Dom}(R_1, \ldots, R_n) = (S_1, \ldots, S_n),$$

$$\text{where } S_i := \mathrm{dom}\Big(P_i, \bigcup_{j \neq i} R_j\Big).$$

The proof uses the Tarski Fixed Point Theorem
plus some geometric argument specific to Euclidean space.

Non-uniqueness example
in Manhattan metric:



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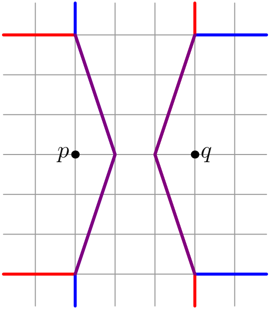
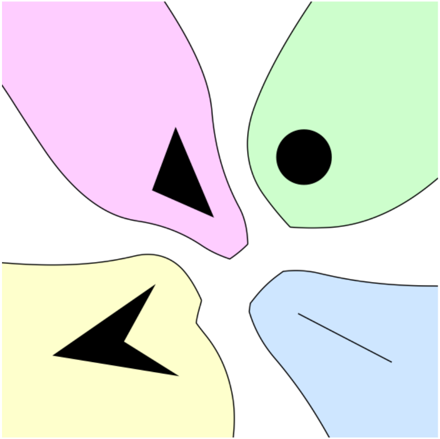


We have shown:

- A k-sector exists.
- The zone diagram exists and is unique.

Problems:

- Is the k-sector unique?
- Properties of the curves
- Algorithm to draw them
- Curves defined by other equations involving dom



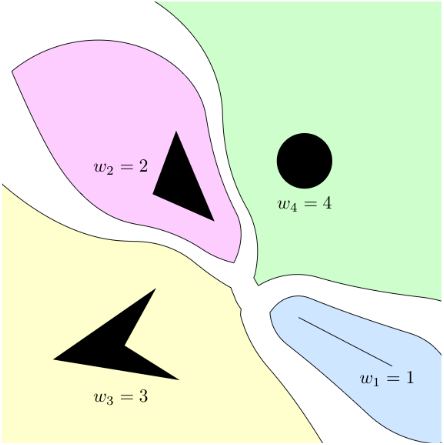
Questions and Answers

Here are my answers to some of the questions asked on- and off-line at the workshop.

Questions and Answers (1)

Have you considered a weighted version of zone diagram?

Not yet.



Questions and Answers (2)

Are the curves algebraic?

We don't know. It is open since [Asano et al. 2006], *even for the trisector between two points.*

Then how did you draw the pictures?

The trisector curve between two points on slide 4 is analytic and we can compute the first few Taylor coefficients [ibid.]. *Other pictures are obtained by iterating* **Dom** *many times.*

Do the boundaries in the zone diagram consist of parts of the trisector curve between two sites?

No [Asano et al. 2007]. *This is an interesting phenomenon not seen in Voronoi diagrams.*

Questions and Answers (3)

I understand that by moving from 2 sites to n sites, bisectors generalize to Voronoi diagrams, and trisectors to zone diagrams. What would be the "k-sector between n objects," then?

We discuss layered zone diagrams in the last section of the extended abstract. They are a possible generalization of k-sectors to n objects, and are defined by equations involving the dom *operator (just as the zone diagram is).*

Can't you prove uniqueness by using the Banach fixed point theorem (a.k.a. contraction principle) with some suitable metric?

We've tried that a hundred times. . . . (Of course there still may be a hundred-and-first metric that works.)

Thanks for the questions!

References

[Asano et al. 2006] T. Asano, J. Matoušek, and T. Tokuyama.
The distance trisector curve.
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[Asano et al. 2007] T. Asano, J. Matoušek, and T. Tokuyama.
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