

# Distance $k$ -Sectors and Zone Diagrams

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## Abstract

We prove the existence and uniqueness of the zone diagram of a given set of sites in Euclidean space. This was known for point sites in the plane, but our proof is simpler even for this specific case. We also show the existence of a distance  $k$ -sector between two sites. Both proofs rely on the Knaster–Tarski theorem on fixed points of monotone functions.

## 1 Introduction

Geometric bisection is a fundamental concept. The points equidistant from given two points lie on a line, and the points equidistant from a point and a line lie on a parabola. The Voronoi diagram, an important structure in computational geometry, can be considered as bisectors generalized to  $n$  points [3, 5].

What happens if bisection is replaced by trisection, or more? Asano et al. [2] introduced the *distance  $k$ -sector* (henceforth just  *$k$ -sector*) by extending the equidistance condition for bisection: the  $k$ -sector of two disjoint closed sets  $P$  and  $Q$  in the Euclidean space  $\mathbb{R}^d$  is a series of  $k - 1$  nonempty sets  $C_1, \dots, C_{k-1} \subseteq \mathbb{R}^d$  such that each  $C_i$  is the bisector of  $C_{i-1}$  and  $C_{i+1}$ , where we regard  $C_0 = P$  and  $C_k = Q$ .

Consider the most basic case where  $P$  and  $Q$  are points in the plane. In this case, the trisector (3-sector) exists and is unique [2]. The existence of a 4-sector of two points is easy: let  $C_2$  be the perpendicular bisector of  $P$  and  $Q$ , and let  $C_1$  and  $C_3$  be the bisecting parabolas. We do not know whether this is the unique solution. Chun et al. [4] have shown that the trisector of a point and a line exists and is unique, and thus a 6-sector of two points exists; see Figure 1. We remark that  $C_1$  and  $C_5$  of this 6-sector are closed curves. Reem and Reich [6] proved that a trisector of given two sets always exists in any metric space. It was conjectured that a  $k$ -sector exists for any  $k$ , but has not been proved even for two points in the plane.

Combining the ideas of Voronoi diagrams and the trisector, we obtain the notion of *zone diagrams* [1]. A zone diagram of nonempty sets (called *sites*)  $P_1, \dots, P_n \subseteq \mathbb{R}^d$  is an  $n$ -tuple  $(R_1, \dots, R_n)$  of subsets of  $\mathbb{R}^d$  satisfying

$$R_i = \text{dom}\left(P_i, \bigcup_{j \neq i} R_j\right), \quad i = 1, \dots, n, \quad (1)$$

where, for sets  $A, B \subseteq \mathbb{R}^d$  (not both empty),  $\text{dom}(A, B) = \{x \in \mathbb{R}^d : d(x, A) \leq d(x, B)\}$  is the *dominance region* of  $A$  over  $B$ , with  $d(X, Y) = \inf_{x \in X, y \in Y} |x - y| \in [0, +\infty]$  denoting the Euclidean distance of sets  $X$  and  $Y$ . Figure 2 shows the zone diagram (and the Voronoi diagram) of line segments

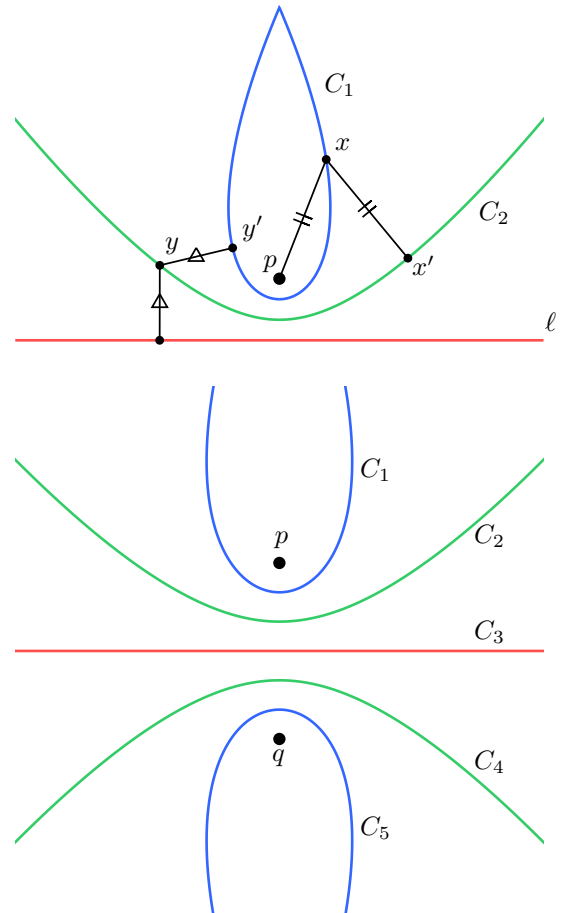


Figure 1: The trisector of line  $l$  and point  $p$  (top) and a 6-sector of points  $p, q$  (bottom).

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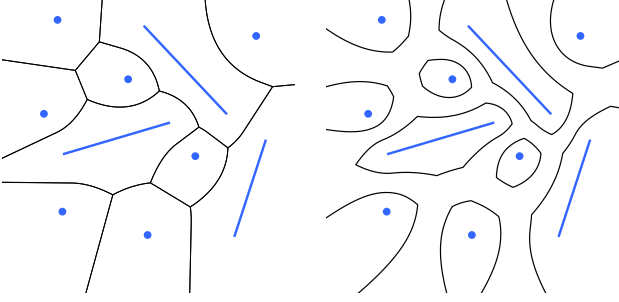


Figure 2: The Voronoi diagram (left) and the zone diagram (right) of points and line segments.

and points in the plane. A zone diagram of two disjoint closed sites gives their trisector. Asano et al. [1] proved the existence and uniqueness of the zone diagram of point sites in the plane, and conjectured that the same is true for general sites and general dimensions. Unfortunately, their proof involves case analysis specific to  $\mathbb{R}^2$  that seems hard to generalize.

The concept of a zone diagram can be immediately generalized to any metric space. Reem and Reich [6] proved the existence of what they call *double zone diagrams*, which are candidates for zone diagrams (see Section 3), in an arbitrary metric space (and even in a still more general setting, which they call *m-spaces*). The existence of a zone diagram is not known in general, and there are counterexamples to uniqueness in some artificial metric spaces [6].

We settle two of the conjectures mentioned above for Euclidean space:

**Theorem 1** *Let  $P_1, \dots, P_n$  be nonempty subsets of  $\mathbb{R}^d$  such that  $d(P_i, P_j) > 0$  for every  $i \neq j$ . Then the zone diagram of  $(P_1, \dots, P_n)$  exists and is unique.*

**Theorem 2** *For any disjoint nonempty closed sets  $P$  and  $Q \subseteq \mathbb{R}^d$ , there exists a  $k$ -sector of  $P$  and  $Q$ .*

These will be proved in Sections 3 and 4.1, respectively. The *Knaster–Tarski fixed point theorem* (Section 2), used already in [6], will play an essential role.

Theorem 1 is new even for the case  $n = 2$ , since uniqueness was known only when the sites are points. The proof is simpler than the ones in [2, 1] even for the case of point sites in the plane.

Given a set  $R \subseteq \mathbb{R}^d$ , we write  $\bar{R}$ ,  $\partial R$  and  $R^c$  for its closure, boundary and complement.

## 2 The Knaster–Tarski fixed point theorem

Fix  $n \in \mathbb{N}$  and let  $\mathcal{L}$  be the set of all ordered  $n$ -tuples of subsets of  $\mathbb{R}^d$ . Given  $\mathbf{A} = (A_1, \dots, A_n) \in \mathcal{L}$  and  $\mathbf{B} = (B_1, \dots, B_n) \in \mathcal{L}$ , define  $\mathbf{A} \leq \mathbf{B}$  if  $A_i \subseteq B_i$  for each  $i$ . This gives a partial ordering on  $\mathcal{L}$ . A function  $g$  from  $\mathcal{S} \subseteq \mathcal{L}$  to  $\mathcal{S}$  is *monotone* (resp. *anti-monotone*)

if  $g(\mathbf{A}) \leq g(\mathbf{B})$  (resp.  $g(\mathbf{B}) \leq g(\mathbf{A})$ ) for any  $\mathbf{A} \leq \mathbf{B}$ . An  $n$ -tuple  $\mathbf{D} \in \mathcal{L}$  is an *upper* (resp. a *lower*) *bound* of  $\mathcal{Z} \subseteq \mathcal{L}$  if  $\mathbf{X} \leq \mathbf{D}$  (resp.  $\mathbf{D} \leq \mathbf{X}$ ) for any  $\mathbf{X} \in \mathcal{Z}$ . A set  $\mathcal{S} \subseteq \mathcal{L}$  is called a *complete lattice* if any subset  $\mathcal{Z} \subseteq \mathcal{S}$  has the least upper bound  $\bigvee \mathcal{Z}$  and the greatest lower bound  $\bigwedge \mathcal{Z}$  in  $\mathcal{S}$ . Following Reem and Reich [6], we will use the Knaster–Tarski Theorem:

**Theorem 3 ([7])** *If  $\mathcal{S}$  is a complete lattice and  $g: \mathcal{S} \rightarrow \mathcal{S}$  is monotone, then  $\mathbf{R} = \bigwedge \{ \mathbf{Y} \in \mathcal{S} : g(\mathbf{Y}) \leq \mathbf{Y} \}$  and  $\mathbf{S} = \bigvee \{ \mathbf{Y} \in \mathcal{S} : g(\mathbf{Y}) \geq \mathbf{Y} \}$  are fixed points of  $g$ . Moreover,  $\mathbf{R} \leq \mathbf{X} \leq \mathbf{S}$  for any fixed point  $\mathbf{X}$ .*

In our applications, the monotonicity required in the theorem is based on the following simple observation [1, 6].

**Lemma 4** *For  $X \subseteq X'$ , we have  $\text{dom}(X, B) \subseteq \text{dom}(X', B)$  and  $\text{dom}(A, X) \supseteq \text{dom}(A, X')$ .*

## 3 Existence and uniqueness of zone diagrams

Let  $\mathcal{S}$  be the complete lattice consisting of all  $n$ -tuples of subsets of  $\mathbb{R}^d$ . Let us fix an  $n$ -tuple  $\mathbf{P} = (P_1, \dots, P_n) \in \mathcal{S}$  of nonempty sets (the sites). Let  $\mathbf{Dom}: \mathcal{S} \rightarrow \mathcal{S}$  be given by  $\mathbf{Dom}(\mathbf{D}) = \mathbf{E}$ , where  $\mathbf{D} = (D_1, \dots, D_n)$  and  $\mathbf{E} = (E_1, \dots, E_n)$  with

$$E_i = \text{dom}\left(P_i, \bigcup_{j \neq i} D_j\right), \quad i = 1, \dots, n. \quad (2)$$

A zone diagram of  $\mathbf{P}$  is exactly a fixed point of  $\mathbf{Dom}$  [1].

The function  $\mathbf{Dom}$  is anti-monotone (Lemma 4), hence  $\mathbf{Dom}^2 = \mathbf{Dom} \circ \mathbf{Dom}$  is monotone. Reem and Reich [6] call a fixed point of  $\mathbf{Dom}^2$  a *double zone diagram* of  $\mathbf{P}$ . From Theorem 3 one obtains the following:

**Theorem 5 ([6, Theorem 5.5])** *The function  $\mathbf{Dom}^2$  has fixed points  $\mathbf{R}$  and  $\mathbf{S}$  such that  $\mathbf{R} = \mathbf{Dom} \mathbf{S}$ ,  $\mathbf{S} = \mathbf{Dom} \mathbf{R}$  and  $\mathbf{R} \leq \mathbf{D} \leq \mathbf{S}$  for any fixed point  $\mathbf{D}$  of  $\mathbf{Dom}^2$ .*

Using this, we will now prove Theorem 1. We remark that the same proof works also for infinitely many sites, every two of them at distance at least  $\varepsilon$ , for some fixed  $\varepsilon > 0$ .

We may assume that the sites  $P_1, \dots, P_n$  are closed (since a zone diagram of their closures is also a zone diagram of the  $P_i$ ).

**Lemma 6** *Let  $P_1, \dots, P_n$  be closed and let  $\varepsilon$  be as above. Suppose that  $n$ -tuples  $\mathbf{D}$  and  $\mathbf{E}$  of subsets of  $\mathbb{R}^d$  satisfy  $\mathbf{D} = \mathbf{Dom} \mathbf{E}$  and  $\mathbf{E} = \mathbf{Dom} \mathbf{D}$ . Let  $i \in \{1, \dots, n\}$  and suppose that  $p$  is the closest point to  $a \in D_i$  in  $P_i$ . Then the convex hull of  $K \cup \{a\}$  is contained in  $D_i$ , where  $K$  is the closed ball with centre  $p$  and radius  $\varepsilon/4$ . (Proof omitted.)*

To prove Theorem 1, it suffices to show that the fixed points  $\mathbf{R} = (R_1, \dots, R_n)$  and  $\mathbf{S} = (S_1, \dots, S_n)$  in Theorem 5 coincide. Suppose, for contradiction, that there are  $i_0$  and  $b_0 \in S_{i_0} \setminus R_{i_0}$ . We define index  $i_t$  and points  $b_t, p_t, a_t$  for each  $t \in \mathbb{N}$  inductively as follows (Figure 3). Suppose that  $i_t$  and  $b_t$  have been defined. Let  $p_t$  be one of the closest points in  $P_{i_t}$  to  $b_t$ , and let  $a_t$  be the unique point (by Lemma 6) where the line segment  $b_t p_t$  meets  $\partial R_{i_t}$ . Since  $a_t \in \partial R_{i_t}$  and  $\mathbf{R} = \mathbf{Dom} \mathbf{S}$ , there are  $i_{t+1} \neq i_t$  and  $b_{t+1} \in S_{i_{t+1}}$  with  $|a_t - b_{t+1}| = |a_t - p_t|$ ; choose any such.

For each  $t \in \mathbb{N}$ , let  $r_t = |a_t - p_t|$ ,  $s_t = |b_t - p_t|$  and  $\theta_t = \angle p_t a_t b_{t+1}$ . Since  $|a_{t+1} - b_t| \geq s_t$  by  $a_{t+1} \in R_{i_{t+1}}$ ,  $b_t \in S_{i_t}$  and  $\mathbf{S} = \mathbf{Dom} \mathbf{R}$ , we have

$$\begin{aligned} s_{t+1} - r_{t+1} &= |a_{t+1} - b_{t+1}| \geq |a_{t+1} - b_t| - |b_t - b_{t+1}| \\ &= |a_{t+1} - b_t| - \sqrt{|a_t - b_t|^2 + |a_t - b_{t+1}|^2} \\ &\quad + 2|a_t - b_t| \cdot |a_t - b_{t+1}| \cos \theta_t \\ &\geq s_t - \sqrt{(s_t - r_t)^2 + r_t^2 + 2(s_t - r_t)r_t \cos \theta_t} \\ &\geq \frac{r_t(s_t - r_t)}{s_t}(1 - \cos \theta_t). \end{aligned} \quad (3)$$

Let  $h$  be the distance from  $b_{t+1}$  to the convex hull of Lemma 6 with  $i = i_t$  and  $a = a_t$ . Since  $\mathbf{S} = \mathbf{Dom} \mathbf{R}$  and  $b_{t+1} \in S_{i_{t+1}}$ , we have

$$\begin{aligned} s_{t+1} \leq h &= r_t \sin \min \left\{ \frac{\pi}{2}, \theta_t - \arcsin \frac{\varepsilon}{4r_t} \right\} \\ &\leq r_t \sin \min \left\{ \frac{\pi}{2}, \theta_t - \gamma \right\}, \end{aligned} \quad (4)$$

where  $\gamma = \arcsin(\varepsilon/4r_0) \leq \theta_t$ . The last inequality is because the  $r_t$  are decreasing.

Let  $\lambda \in (0, 1)$  be small enough that  $(1 - \cos \gamma)^\lambda > \cos(\gamma/2)$ . By (3) and (4), we have

$$\frac{(s_{t+1} - r_{t+1})^\lambda}{s_{t+1}} \geq \frac{(s_t - r_t)^\lambda}{r_t^{1-\lambda} s_t^\lambda} B \geq \frac{(s_t - r_t)^\lambda}{s_t} B, \quad (5)$$

where

$$\begin{aligned} B &= \frac{(1 - \cos \theta_t)^\lambda}{\sin \min \{ \pi/2, \theta_t - \gamma \}} \\ &\geq \begin{cases} \frac{(1 - \cos \gamma)^\lambda}{\cos(\gamma/2)} & \text{if } \theta_t < \frac{\pi}{2} + \frac{\gamma}{2}, \\ \frac{(1 + \sin(\gamma/2))^\lambda}{\sin(\pi/2)} & \text{otherwise} \end{cases} \\ &\geq \min \left\{ \frac{(1 - \cos \gamma)^\lambda}{\cos(\gamma/2)}, (1 + \sin(\gamma/2))^\lambda \right\} > 1. \end{aligned} \quad (6)$$

Thus,  $B$  is bounded from below by a constant exceeding 1 independently of  $t$ . Therefore, (5) implies that  $(s_t - r_t)^\lambda / s_t$  is unbounded as  $t \in \infty$ , contradicting  $(s_t - r_t)^\lambda / s_t \leq s_t^{-1+\lambda} \leq (\varepsilon/4)^{-1+\lambda}$ . We have proved Theorem 1.

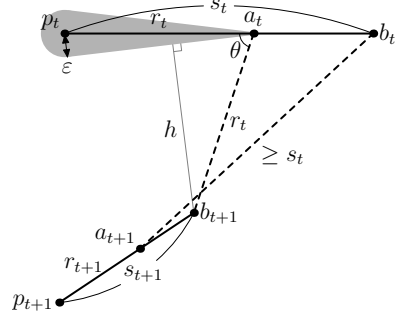


Figure 3: The shaded region is in  $R_{i_t}$  by Lemma 6.

#### 4 Distance $k$ -sectors

The *bisector* of nonempty subsets  $A$  and  $B$  of  $\mathbb{R}^d$  is defined by

$$\text{bisect}(A, B) = \{ z \in \mathbb{R}^d : d(z, A) = d(z, B) \}. \quad (7)$$

It is not hard to see that

$$\text{bisect}(A, B) = \partial \text{dom}(A, B) = \partial \text{dom}(B, A) \quad (8)$$

if the closures of  $A$  and  $B$  are disjoint. Let  $P$  and  $Q$  be disjoint nonempty closed subsets of  $\mathbb{R}^d$ . A  $k$ -sector of  $P$  and  $Q$  is a sequence  $(C_1, \dots, C_{k-1})$  of nonempty subsets of  $\mathbb{R}^d$  satisfying

$$C_i = \text{bisect}(C_{i-1}, C_{i+1}), \quad i = 1, \dots, k-1, \quad (9)$$

where  $C_0 = P$  and  $C_k = Q$ .

##### 4.1 Existence

Similarly to Section 2, let  $\mathcal{L}$  be the complete lattice of all  $(k-1)$ -tuples of subsets of  $\mathbb{R}^d$  (ordered by componentwise inclusion). For  $\mathbf{D} = (D_1, \dots, D_{k-1}) \in \mathcal{L}$  we define  $F(\mathbf{D}) = (E_1, \dots, E_{k-1})$  by

$$E_i = \text{dom}(D_{i-1} \cup P, D_{i+1}^c \cup Q), \quad i = 1, \dots, k-1, \quad (10)$$

where  $D_0 = P$  and  $D_k = Q^c$ . Since  $F$  is monotone by Lemma 4, it has a fixed point  $(R_1, \dots, R_{k-1})$  by Theorem 3. The following lemma says that a fixed point consists of a hierarchy of sets  $P = R_0 \subseteq R_1 \subseteq \dots \subseteq R_n = Q$  with separated boundaries.

**Lemma 7** *If  $(R_1, \dots, R_{k-1})$  is a fixed point of the function  $F$  above, then  $R_i \cap \overline{R_j^c} = \emptyset$  for each  $i$  and  $j$  with  $0 \leq i < j \leq k$ . (Proof omitted.)*

We are now ready to prove Theorem 2. Let  $(R_1, \dots, R_{k-1})$  be a fixed point as above. Let  $C_i = \partial R_i$  for each  $i$ . Then (9) is proved as follows:

$$\begin{aligned} C_i &= \partial \text{dom}(R_{i-1} \cup P, R_{i+1}^c \cup Q) \\ &= \partial \text{dom}(R_{i-1}, R_{i+1}^c) \\ &= \text{bisect}(R_{i-1}, R_{i+1}^c) = \text{bisect}(C_{i-1}, C_{i+1}). \end{aligned} \quad (11)$$

The third equality is by (8) using Lemma 7. The last equality is because  $d(a, X) = d(a, \partial X)$  for  $a \notin X$ .

## 4.2 The least and the greatest solutions

We conjecture that the  $k$ -sector of two disjoint closed sites  $P, Q \subset \mathbb{R}^d$  is always unique. Here we present some supporting evidence.

Let  $\mathcal{L}$  and  $F$  be as above. Let  $\perp = (\emptyset, \dots, \emptyset)$  be the least element of  $\mathcal{L}$ . For  $\mathbf{D} \in \mathcal{L}$  satisfying  $\mathbf{D} \leq F(\mathbf{D})$ , denote by  $F^\infty(\mathbf{D})$  the componentwise closure of  $\bigvee \{F^n(\mathbf{D}) : n \in \mathbb{N}\}$ .

**Lemma 8**  $F^\infty(\perp)$  is a fixed point of  $F$ . (Proof omitted.)

In fact,  $F^\infty(\perp)$  is the least fixed point, because by monotonicity,  $F^\infty(\perp) \leq F^\infty(\mathbf{D}) = \mathbf{D}$  for any fixed point  $\mathbf{D}$ . We can prove similarly that  $F^\infty(\top) = \bigwedge \{F^n(\top) : n \in \mathbb{N}\}$  is the greatest fixed point, where  $\top = (\mathbb{R}^d, \dots, \mathbb{R}^d)$ . This gives, in this particular setting, a “constructive” description of the smallest and largest fixed points from Theorem 3. Thus the fixed point of  $F$  is unique if  $F^\infty(\perp) = F^\infty(\top)$ .

Using this, we computed, for point sites  $P = \{(0, 1)\}$  and  $Q = \{(0, -1)\}$ , a lower bound of  $F^\infty(\perp)$  for each  $k \leq 11$  by iteratively applying  $F$  on the pixel grid, each time underestimating the sets. We did this experiment with several different pixel sizes, and it seemed that the picture always eventually becomes symmetric with respect to the  $x$ -axis up to a few pixels, except near the edges of the grid. Thus, even a lower bound of the least fixed point  $F^\infty(\perp)$  is nearly symmetric; this implies that  $F^\infty(\perp)$  and  $F^\infty(\top)$  (which must be symmetric to each other) are very close, although we do not have a proof that they coincide.

## 5 Layered zone diagrams

Zone diagrams and  $k$ -sectors were defined by simple equations involving the dom operator. We can consider various similar systems of equations. For example, given sites  $P_1, \dots, P_n$ , consider the equations

$$R_i = \text{dom}\left(T_i, \bigcup_{j \neq i} T_j\right), \quad T_i = \text{dom}(P_i, R_i^c) \quad (12)$$

for  $i = 1, \dots, n$ . These equations specify the relation that the layered zones  $R_i \supseteq T_i \supseteq P_i$  should satisfy. They imply that, between sets  $P_i$  and  $P_j$  that are located close to each other, there are four regions bounded by their 4-sector. Thus, a solution to (12) can be regarded as a generalization to  $k = 4$  of Voronoi diagrams (which correspond to  $k = 2$ ) and of zone diagrams (which correspond to  $k = 3$ ). We thus call a solution to (12) a *layered zone diagram* of the  $P_i$  (with four layers in this case).

The whole space is divided into  $R_1, \dots, R_n$ . One might perhaps expect that  $R_i$  is the Voronoi region of  $P_i$  among the sites (which is true for the case of two

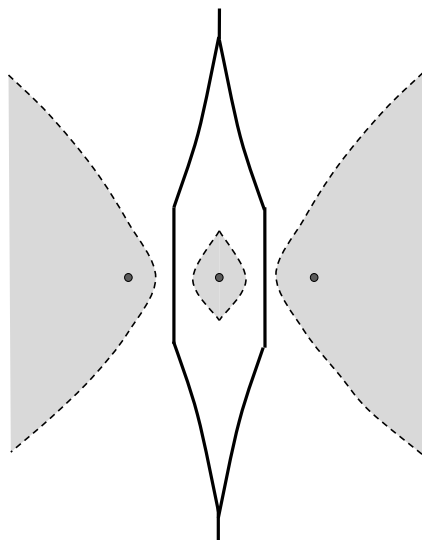


Figure 4: A layered zone diagram on three point sites. The shaded regions are the  $T_i$ .

point sites, where we deal with a 4-sector), but it is not: For  $P_1 = \{(-1, 0)\}$ ,  $P_2 = \{(0, 0)\}$ ,  $P_3 = \{(1, 0)\}$ , the Voronoi region of  $P_2$  is a vertical strip, but it can be proved that  $R_2$  is bounded (Figure 4).

Currently we do not have a proof of existence of such layered zone diagrams. But it is easy to show, using Theorem 3 as usual, that an appropriate “double layered zone diagram” exists, and this is a natural candidate for a layered zone diagram.

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