Differential Recursion

AKITOSHI KAWAMURA
University of Toronto

We present a redevelopment of the theory of real-valued recursive functions that was introduced by C. Moore in 1996 by analogy with the standard formulation of the integer-valued recursive functions. While his work opened a new line of research on analog computation, the original paper contained some technical inaccuracies. We discuss possible attempts to remove the ambiguity in the behaviour of the operators on partial functions, with a focus on his "primitive recursive" functions generated by the *differential recursion* operator that solves initial value problems. Under a reasonable reformulation, the functions in this class are shown to be analytic and computable in a strong sense in Computable Analysis. Despite this well-behavedness, the class turns out to be too big to have the originally purported relation to differentially algebraic functions, and hence to C. E. Shannon's model of analog computation.

Categories and Subject Descriptors: F.1.1 [**Computation by Abstract Devices**]: Models of Computation—*Relations between models*

General Terms: Theory

Additional Key Words and Phrases: analog computation, differentially algebraic functions, initial value problems, real recursive functions, transcendentally transcendental functions

1. INTRODUCTION

There are several kinds of theoretical models that discuss "computability" and "complexity" of real functions. *Computable Analysis* [Grzegorczyk 1955; Weihrauch 2000] uses approximation to bring real numbers into the framework of the standard Computability Theory that deals with discrete data in discrete time. Another approach uses the *algebraic models*, such as the one by Blum et al. [1997], in which continuous quantities are treated as entities in themselves but the machine still works with discrete clock ticks.

A third approach is *analog computation* in which not only are the data real-valued, but also the transition takes place in continuous time [Orponen 1997; Bournez and Campagnolo 2008]. One of the oldest and the best studied models of such computation is the *General Purpose Analog Computer* [Shannon 1941]

Author's address: Department of Computer Science, University of Toronto, 10 King's College Road, Toronto, ON, Canada M5S 3G4.

Parts of this work were presented at the Sixth International Workshop on Computability and Complexity in Analysis (CCA 2004) [Kawamura 2005] and the Second Conference on Computability in Europe (CiE 2006). Supported in part by the International Information Science Foundation in Japan, the Japan Society for Promotion of Science (Research Fellowship, DC1 18-11700), the Nakajima Foundation, and the Natural Sciences and Engineering Research Council of Canada.

that models the *Differential Analyzer* [Bush 1931], a computing device built and put to use during the thirties through the fifties. The GPAC, after some refinements [Pour-El 1974; Lipshitz and Rubel 1987; Graça 2004], was shown to generate (in a sense) all and only the *differentially algebraic* functions. We will explore this class in Section 2.

Little is known about how such analog models relate to the standard (digital) computability. Moore [1996] addressed this question for his new function classes that also try to express the power of GPAC-like computation. In imitation of Kleene's characterization of the usual recursive functions, these classes are defined as the closures under the operators of *differential recursion* and *zero-finding*, which are supposedly real-number versions of primitive recursion and minimization. He makes the following claims, among others, that relate his classes of *real primitive recursive* and *real recursive* functions to analog and digital computation, respectively [Moore 1996, Propositions 9 and 13].

CLAIM 1.1. *Real primitive recursive functions are differentially algebraic.*[1]

CLAIM 1.2. *Each (partial) recursive function on the nonnegative integers (in the standard sense) is a restriction of some real recursive function.*

Although his work has inspired numerous subsequent studies on the classes and their variants [Campagnolo 2001; Campagnolo and Moore 2001; Graça 2004; Mycka and Costa 2004; Bournez and Hainry 2006; Loff et al. 2007; Campagnolo and Ojakian 2008], it lacked formality in some ways, as already pointed out [Campagnolo et al. 2000; Graça 2002]. In fact, the definition of the classes is somewhat ambiguous. In Section 3 of the present paper, we reformulate Moore's theory up to the real primitive recursive functions in a mathematically rigorous way. It turns out that with a reasonable modification on the definition, the real primitive recursive functions are analytic, computable and have computable domain in one of the senses in Computable Analysis. Despite this well-behavedness, it is shown in Section 4 that the class is too large to satisfy Claim 1.1. One of our counterexamples gives an alternative proof of the result [Katriel 2003] on one of Rubel's problems about algebraic differential equations [Rubel 1992, Problem 31]. In Section 5, we discuss some issues about classes other than the real primitive recursive functions, including Claim 1.2.

We write $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ for the sets of nonnegative integers (including 0), integers and real numbers, respectively.

Partial functions. In this paper, a *function* $f : \mathbf{R}^m \rightharpoonup \mathbf{R}^n$ may be *partial*, as opposed to *total*; that is, the value $f\,x \in \mathbf{R}^n$ may be undefined for some points $x \in \mathbf{R}^m$. The set of x for which $f\,x$ is defined is called the *domain* of f and denoted by $\operatorname{dom} f$. By the *restriction* of f to set $J \subseteq \mathbf{R}^m$ we mean the function g with $\operatorname{dom} g = J \cap \operatorname{dom} f$ such that $g\,x = f\,x$ for every $x \in \operatorname{dom} g$. When $\operatorname{dom} f$ is open, f is said to be *analytic* if for every $a = (a_0, \dots, a_{m-1}) \in \operatorname{dom} f$ there are an open set $J \subseteq \operatorname{dom} f$ containing a and a family $(c_p)_{p \in \mathbf{N}^m}$ of n-tuples of real numbers such that

[1] Moore writes M_0 for the class of real primitive recursion functions. Claim 1.1 was later replaced by a similar claim [Campagnolo et al. 2000, Proposition 2] for a more "restricted" class $\mathcal{G}$ than M_0, but its definition is again unclear.

the sum of $c_p \cdot (x_0 - a_0)^{p_0} \cdots (x_{m-1} - a_{m-1})^{p_{m-1}}$ over all $p = (p_0, \ldots, p_{m-1}) \in \mathbf{N}^m$ converges to fx for each $x = (x_0, \ldots, x_{m-1}) \in J$ (regardless of the order in which the summation is taken). When f is analytic, we write $\mathrm{D}^{(a_0,\ldots,a_{m-1})} f$ (and not $\partial^{a_0+\cdots+a_{m-1}} f / \partial t_0^{a_0} \cdots \partial t_{m-1}^{a_{m-1}}$) for the mixed partial derivative of f of order a_i along the i^{th} place (which is known to exist). Properties of analytic functions are well known [Krantz and Parks 2002, Chapters 1 and 2], but we need to be careful in restating them for partial functions. For example, the connectedness assumption cannot be dropped in the following *Identity Theorem*.

THEOREM 1.3. *An analytic function with open connected domain that vanishes on a nonempty open set vanishes everywhere.*

Moore [1996] does not pay explicit attention to partial functions. We believe that this is responsible for the ambiguous claims made in his seminal work as well as in some of the subsequent works by other authors. Although there are some situations in mathematical analysis where we can pretend that the functions involved are total (namely, when we are only discussing properties defined *locally*, such as continuity or analyticity), this is not the case with the notions we want to discuss here. If, say, the above Claim 1.2 is to make any nontrivial sense, it is clearly inappropriate to talk about "real recursiveness at x," as there is a real function which is simple locally but the restriction of which to $\mathbf{N}$ is highly complex in the recursion-theoretic sense. We therefore emphasize that partial functions must be dealt with seriously, and devote this paper to accordingly reformulating the theory wherever possible.

2. DIFFERENTIALLY ALGEBRAIC FUNCTIONS

This section shows some properties of *differentially algebraic functions*. Roughly speaking, they are defined to be analytic functions whose derivatives satisfy a nontrivial polynomial relation. For example, the sine function is differentially algebraic because $\sin t + \mathrm{D}^2 \sin t = 0$, while Euler's gamma function (see Section 4) satisfies no such relation [Hölder 1886]. We begin by establishing the equivalence of several variants of the definition.

THEOREM 2.1. *Let m and $i < m$ be nonnegative integers and $f : \mathbf{R}^m \rightharpoonup \mathbf{R}$ be an analytic function with open domain. Consider the following statements:*

(i) *for any open connected set $J \subseteq \operatorname{dom} f$, there is a $\mathbf{Z}$-coefficient nonzero polynomial P such that*

$$P\big(fx, \mathrm{D}^{e_i} fx, \mathrm{D}^{2\cdot e_i} fx, \ldots, \mathrm{D}^{(\text{arity } P - 1)\cdot e_i} fx\big) = 0 \tag{1}$$

for all $x \in J$, where $e_i \in \mathbf{N}^m$ is the vector whose i^{th} component is 1 and others are 0;

(ii) *for each $\hat{x} \in \operatorname{dom} f$, there are a $\mathbf{Z}$-coefficient nonzero polynomial P and an open set J containing $\hat{x}$ such that we have (1) for all $x \in J$;*

(iii) *for each $\hat{x} \in \operatorname{dom} f$, there are a $\mathbf{Z}$-coefficient nonzero polynomial P and an open interval J containing the i^{th} component of $\hat{x}$ such that we have (1) for all x whose i^{th} component is in J and whose other components equal those of $\hat{x}$;*

(iv) *for each $x \in \operatorname{dom} f$, there is a $\mathbf{Z}$-coefficient nonzero polynomial P with (1).*

Let $(\mathrm{i}_{\mathbf{R}})$*,* $(\mathrm{ii}_{\mathbf{R}})$ *and* $(\mathrm{iii}_{\mathbf{R}})$ *be the statements obtained by replacing* $\mathbf{Z}$ *by* $\mathbf{R}$ *in* (i)*,* (ii) *and* (iii)*, respectively. Then* (i)*,* (ii)*,* (iii)*,* (iv)*,* $(\mathrm{i}_{\mathbf{R}})$*,* $(\mathrm{ii}_{\mathbf{R}})$ *and* $(\mathrm{iii}_{\mathbf{R}})$ *are equivalent.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ $(\mathrm{i}_{\mathbf{R}})$ $\Rightarrow$ $(\mathrm{ii}_{\mathbf{R}})$ $\Rightarrow$ $(\mathrm{iii}_{\mathbf{R}})$ are obvious. See Theorem A.1 in the appendix for $(\mathrm{iii}_{\mathbf{R}})$ $\Rightarrow$ (iii). To see (iv) $\Rightarrow$ (i), take an open connected set $J \subseteq \operatorname{dom} f$ and consider, for each $\mathbf{Z}$-coefficient polynomial P, the set J_P of all $x \in J$ satisfying (1). Since by (iv) these countably many closed sets J_P cover J, one of them must have nonempty interior by Baire Category Theorem A.2. This J_P must then equal J by the Identity Theorem 1.3. □

Definition 2.2. Let m and n be nonnegative integers. An analytic function $f : \mathbf{R}^m \rightharpoonup \mathbf{R}$ with open domain is *differentially algebraic*[2] if for each $i < m$ it satisfies one, and hence all, of the clauses in Theorem 2.1. An analytic function $f : \mathbf{R}^m \rightharpoonup \mathbf{R}^n$ with open domain is *differentially algebraic* if for each $j < n$, the function taking $x \in \operatorname{dom} f$ to the j^{th} component of $f x$ is differentially algebraic.

Note that f need not be the *unique* solution of (1). For example, every function (with open domain) that is constant on each connected component of its domain is differentially algebraic because of the single set of equations $\mathrm{D}^{e_i} f x = 0$.

The clauses (ii)–(iv) show that being differentially algebraic is a "local" property.

When $\operatorname{dom} f$ is connected, (i) reduces to the following statement:

(i′) there is a $\mathbf{Z}$-coefficient nonzero polynomial P that satisfies (1) for all $x \in \operatorname{dom} f$.

Hence, (iii) shows that, as long as $\operatorname{dom} f$ is connected, our definition is equivalent to that of most authors, including Moore [1996], who first state the definition for $m = 1$ by (i′) and then extend it to $m > 1$ by saying that a function is differentially algebraic when it is so as a unary function of each argument when all other arguments are held fixed.[3] We proved Theorem 2.1 because the characterization (i) for $m > 1$ will play an essential role in disproving Claim 1.1 in Section 4.

We omit the easy differential equations that show the following lemma.

Lemma 2.3. *The following functions are differentially algebraic: for each* $i, m \in \mathbf{N}$ *with* $i < m$*, the* m*-ary projection* $\mathrm{id}_i^{m \to 1}$ *to the* i^{th} *component; binary addition* $+$ *and multiplication* $\cdot$*; functions* inv_+ *(taking* $x > 0$ *to* $1/x$*),* sqrt_+ *(taking* $x > 0$ *to* $\sqrt{x}$*) and* $\ln$ *(natural logarithm) defined on* $(0, \infty)$*; total functions* $\sin$*,* $\cos$ *and* $\exp$*.*

Let us characterize differentially algebraic functions in yet another way when $n = 1$. For field E, its subfield F and set $B \subseteq E$, we write $F(B)$, agreeing tacitly on E, for the smallest subfield of E that includes F and B. We write $\overline{F}$ for the set of elements of E at which some F-coefficient unary nonzero polynomial vanishes.

Let $J \subseteq \mathbf{R}^m$ be open and consider the ring $\mathrm{C}^\omega[J]$ of analytic functions $g : \mathbf{R}^m \rightharpoonup \mathbf{R}$ with domain J. Note that $\mathbf{R}$ is embedded into this ring by regarding each real number as a constant function. To assert (1) for all $x \in J$ is to say in $\mathrm{C}^\omega[J]$ that

$$P\big(f, \mathrm{D}^{e_i} f, \mathrm{D}^{2 \cdot e_i} f, \ldots, \mathrm{D}^{(\operatorname{arity} P - 1) \cdot e_i} f\big) = 0. \tag{2}$$

[2] Also termed *algebraically transcendental* or *hypotranscendental*. Functions *without* this property is said to be *transcendentally transcendental* or *hypertranscendental*. Equations of form (1) are often called *algebraic differential equations*.

[3] Their definition for $m = 1$ still differs from (i′) in that it replaces (1) by $P(x, f x, \mathrm{D}^{e_i} f x, \ldots, \mathrm{D}^{(\operatorname{arity} P - 2) \cdot e_i} f x) = 0$. But the proof of (a) $\Rightarrow$ (b) of Lemma 2.4 shows that this difference is superficial.

If J is connected, $\mathrm{C}^\omega[J]$ has the fraction field by Theorem 1.3, so the notation $\mathbf{R}(\cdot)$ in the following lemma makes sense. We write $\mathbf{D}\,f = \{\, \mathrm{D}^a f \mid a \in \mathbf{N}^m \,\}$.

LEMMA 2.4. *Let $J \subseteq \mathbf{R}^m$ be open and connected. For $f \in \mathrm{C}^\omega[J]$, the following are equivalent:*

(a) *f is differentially algebraic;*

(b) *$\mathbf{D}\,f \subseteq \mathbf{R}(B)$ for some finite set $B \subseteq \mathbf{D}\,f$;*

(c) *$\mathbf{D}\,f \subseteq \overline{\mathbf{R}(B)}$ for some finite set $B \subseteq \mathbf{R}(\mathbf{D}\,f)$.*

PROOF. The implication (b) $\Rightarrow$ (c) is trivial. The Transcendence Degree Theorem A.3 shows (c) $\Rightarrow$ (a). For (a) $\Rightarrow$ (b), assume that for each i we have an $\mathbf{R}$-coefficient polynomial P_i in $N_i + 1$ variables such that

$$P_i(f, \mathrm{D}^{e_i} f, \mathrm{D}^{2\cdot e_i} f, \ldots, \mathrm{D}^{N_i \cdot e_i} f) = 0. \tag{3}$$

By choosing N_i to be smallest and then the degree of P_i in the last argument to be smallest, we may assume that

$$\varXi = (\mathrm{D}^{(0,\ldots,0,1)} P_i)(f, \mathrm{D}^{e_i} f, \mathrm{D}^{2\cdot e_i} f, \ldots, \mathrm{D}^{N_i \cdot e_i} f) \tag{4}$$

is nonzero, where $\mathrm{D}^{(0,\ldots,0,1)} P_i$ denotes tha "formal" derivative of P_i along its last argument. Define partial order $\leq$ on $\mathbf{N}^m$ by componentwise domination. We show by induction on $a \in \mathbf{N}^m$ with respect to $\leq$ that $\mathrm{D}^a f$ belongs to $\mathbf{R}(\{\, \mathrm{D}^b f \mid b \leq (N_0, \ldots, N_{m-1}) \,\})$. We may assume that $a \geq N_i \cdot e_i$ for some i, since otherwise the claim is trivial. Apply $\mathrm{D}^{a - N_i \cdot e_i}$ to each side of (3) and use the chain rules to see that $\varPsi + \varXi \cdot \mathrm{D}^a f = 0$ for some polynomial $\varPsi$ in derivatives of f of order strictly $\leq a$. Since these derivatives enjoy the induction hypothesis, so does $\mathrm{D}^a f = -\varPsi/\varXi$. □

Apart from purely theoretical interest, the significance of differentially algebraic functions lies in their relation to the *General Purpose Analog Computer*, a model of analog computation introduced by Shannon [1941] and later refined by Pour-El [1974]. More precisely, if a function $f : \mathbf{R} \rightharpoonup \mathbf{R}$ with nonempty domain is differentially algebraic, then the restriction of f to some nonempty subset of $\operatorname{dom} f$ is GPAC generable [Pour-El 1974, Theorem 4]; conversely, if a function $f : \mathbf{R} \rightharpoonup \mathbf{R}$ with nonempty domain is GPAC generable, then the restriction of f to some nonempty subset of $\operatorname{dom} f$ is differentially algebraic [Lipshitz and Rubel 1987, Theorem 2]. Graça [2004] later introduced the *Polynomial GPAC*, a simpler refinement than Pour-El's, and proved analogous results.

3. REAL PRIMITIVE RECURSIVE FUNCTIONS

The class of real primitive recursive functions is defined as the smallest class containing some basic functions and closed under the three operators to be explained below. Unfortunately, the original definition [Moore 1996] contains ambiguity, resulting in some inconsistent claims about the class. To remedy this, we shall revisit the definitions carefully. Section 3.1 defines the first two of the operators. The third operator DR that solves initial value problems is introduced in Section 3.2, with special care taken with the meaning of "unique" solutions. Section 3.3 shows that, although initial value problems can be unsolvable in general, DR is quite well-behaved in both analytic and computational senses if we restrict its input to real primitive recursive functions. Section 3.4 discusses a variant of DR by Campagnolo.

3.1 Two basic operators

An *operator* is a total mapping that takes functions of a fixed arity to functions of a fixed arity. The real primitive recursive functions are defined through three (families of) operators JX, CM and DR (for *juxtaposition*, *composition* and *differential recursion*). The first two are very simple.

Definition 3.1. For $g_0, \dots, g_{n-1} : \mathbf{R}^m \rightharpoonup \mathbf{R}$, define $h = \textsc{jx}(g_0, \dots, g_{n-1}) : \mathbf{R}^m \rightharpoonup \mathbf{R}^n$ by $\operatorname{dom} h = \operatorname{dom} g_0 \cap \cdots \cap \operatorname{dom} g_{n-1}$ and $hx = (g_0 x, \dots, g_{n-1} x)$.

Definition 3.2. For $f : \mathbf{R}^m \rightharpoonup \mathbf{R}^n$ and $g : \mathbf{R}^l \rightharpoonup \mathbf{R}^m$, define $h = \textsc{cm}(f, g) : \mathbf{R}^l \rightharpoonup \mathbf{R}^n$, also written $f \circ g$, by $\operatorname{dom} h = \{\, x \in \operatorname{dom} g \mid gx \in \operatorname{dom} f \,\}$ and $hx = f(gx)$.

To be strict, we have operators $\textsc{jx}^{m,n}$ and $\textsc{cm}^{l,m,n}$ for each arity, but we prefer brevity and rely on the context. Note that $f \circ (g \circ h) = (f \circ g) \circ h$.

As remarked in Section 1, it is important to define precisely what the operators do on partial functions. Note how the above definitions specify the domain of the functions constructed. If gx is not defined, neither is $(f \circ g)x$, even if, say, f is constant. We thus work in the following (informally stated) general principle.

Principle 3.3. *For the value of an expression to be defined, the value of each of its subexpressions has to be defined.*

We remark that this was not explicitly intended by Moore. In fact, he presents an example to the contrary when he claims [Moore 1996, Section 6] that the total function $\overline{\text{inv}} : \mathbf{R} \rightharpoonup \mathbf{R}$ given by

$$\overline{\text{inv}}\, x = \begin{cases} 0 & \text{if } x = 0, \\ 1/x & \text{otherwise} \end{cases} \tag{5}$$

is obtained by composing the binary multiplication with $\textsc{jx}(\text{zero?}, g)$, where g is the restriction of $\overline{\text{inv}}$ to $\mathbf{R} \setminus \{0\}$ and zero? is the total function that takes 0 to 0 and everything else to 1. Some authors point this out [Campagnolo 2001, p. 22] and criticize it [Graça 2002, p. 47]. Without discussing which definition is "natural," we adopt our restrictive Definitions 3.1 and 3.2, simply because we do not know how to give a general definition that would admit this construction of $\overline{\text{inv}}$.

Analyticity is preserved by JX and CM [Krantz and Parks 2002, Proposition 2.2.8]. Differentially algebraic functions are preserved under JX by definition. The following is essentially due to Ostrowski [1920].

Theorem 3.4. *The property of being differentially algebraic is preserved by* CM.

Proof. It suffices to show that if $f : \mathbf{R}^m \rightharpoonup \mathbf{R}$ and $g_0, \dots, g_{m-1} : \mathbf{R}^l \rightharpoonup \mathbf{R}$ are differentially algebraic, so is $f \circ g$, where $g = \textsc{jx}(g_0, \dots, g_{m-1})$. We may assume that $\operatorname{dom} g$ and $J = \operatorname{dom}(f \circ g)$ are connected. We use characterization (b) of Lemma 2.4. By the chain rule, each element of $\mathbf{D}(f \circ g)$ belongs to

$$\mathbf{R}\big(\{\, d \circ g \mid d \in \mathbf{D}\, f \,\} \cup \bigcup_{i=0}^{m-1} \{\, q{\restriction_J} \mid q \in \mathbf{D}\, g_i \,\}\big), \tag{6}$$

where $q{\restriction_J}$ denotes the restriction of q to J. By the assumption, there are finite subsets $A \subseteq \mathbf{D}\, f$ and $B_i \subseteq \mathbf{D}\, g_i$ with $\mathbf{D}\, f \subseteq \mathbf{R}(A)$ and $\mathbf{D}\, g_i \subseteq \mathbf{R}(B_i)$ for each $i < m$. Therefore, (6) stays unchanged by replacing $\mathbf{D}\, f$ by A and $\mathbf{D}\, g_i$ by B_i. □

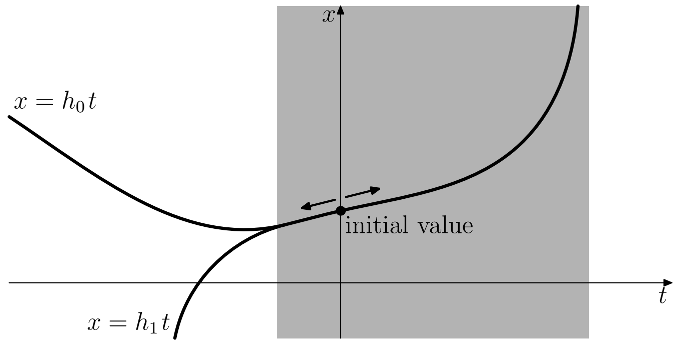

Fig. 1. When the equation is satisfied by both h_0 and h_1 (as well as their restriction to each interval containing the origin), how do we formulate the fact that the shaded interval is the domain of the "unique solution"?

3.2 Definition of differential recursion

The third operator DR is defined to map functions $f : \mathbf{R}^m \rightharpoonup \mathbf{R}^n$ and $g : \mathbf{R}^{m+1+n} \rightharpoonup \mathbf{R}^n$ to the solution $h : \mathbf{R}^{m+1} \rightharpoonup \mathbf{R}^n$ of the integral equation

$$h(v,t) = f\,v + \int_0^t g\big(v, \tau, h(v,\tau)\big) \cdot \mathrm{d}\tau. \tag{7}$$

Moore introduced this operator because it resembles the (discrete) primitive recursion scheme in that the initial value $h(v,0)$ is specified by f and then $h(v,t)$ changes with "time" t according to the function g that refers back to the current value of h. Unlike the discrete case, however, this equation does not always define $h(v,t)$ for all t. For example, the equation $ht = \int_0^t \big(1 + (h\tau)^2\big) \cdot \mathrm{d}\tau$ is satisfied by the tangent function restricted to the open interval $(-\pi/2, \pi/2)$, but by no function with bigger domain. Moore's intention was to pick this "unique" solution on $(-\pi/2, \pi/2)$. But the definition need to be stated carefully, because the restriction of a solution to any subinterval $J \subseteq (-\pi/2, \pi/2)$ around 0 also satisfies the same equation on J. Thus, out of the set K of all solutions, we need to pick one function that deserves to be called the unique solution defined on the largest possible interval (Figure 1). Though Moore did not discuss this issue explicitly, it is not hard to formulate this intuition: for a set K of functions of a type, let K' be the set of $k \in K$ such that the restriction of any function in K to $\operatorname{dom} k$ is a restriction of k. Then any two functions in K' agree on the intersection of their domains. Let uniq_K be the function whose domain is the union of the domains of all functions in K' and of which each function in K' is a restriction.

Definition 3.5. Let $f : \mathbf{R}^m \rightharpoonup \mathbf{R}^n$ and $g : \mathbf{R}^{m+1+n} \rightharpoonup \mathbf{R}^n$. For each $v \in \mathbf{R}^m$, let K_v be the set of all functions $k : \mathbf{R} \rightharpoonup \mathbf{R}^n$ such that

(a) $\operatorname{dom} k$ is either the empty set or a possibly unbounded interval containing 0,

(b) $v \in \operatorname{dom} f$ if $\operatorname{dom} k$ is nonempty,

(c) $(v, \tau, k\tau) \in \operatorname{dom} g$ for each $\tau \in \operatorname{dom} k$, and

(d) for every $t \in \operatorname{dom} k$, we have

$$k t = f v + \int_0^t g(v, \tau, k\tau) \cdot \mathrm{d}\tau. \tag{8}$$

Define $h = \textsc{dr}(f, g) : \mathbf{R}^{m+1} \rightharpoonup \mathbf{R}^n$ by

$$\operatorname{dom} h = \{\, (v, t) \in \mathbf{R}^{m+1} \mid t \in \operatorname{dom} \mathit{uniq}_{K_v} \,\}, \qquad h(v, t) = \mathit{uniq}_{K_v} t. \tag{9}$$

Definition 3.6. The class of *real primitive recursive* functions is the smallest class that contains the three 0-ary functions, each taking values 0, 1 and -1, and is closed under $\textsc{jx}$, $\textsc{cm}$ and $\textsc{dr}$.

Though our initial functions are 0-ary, the n-ary constant functions $0^{n\to1}$, $1^{n\to1}$, $-1^{n\to1}$ can be built by $0^{n\to1} = 0^{0\to1} \circ \textsc{jx}(\,)$, for example, where $\textsc{jx}$ is really $\textsc{jx}^{n,0}$ (see the remark following Definition 3.2).

Lemma 3.7. *The functions in Lemma 2.3 are all real primitive recursive.*

Proof. Let $\mathrm{id}_i^{i+1\to1} = \textsc{dr}(0^{i\to1}, 1^{i+2\to1})$ and $\mathrm{id}_i^{n+1\to1} = \textsc{dr}(\mathrm{id}_i^{n\to1}, 0^{n+2\to1})$ inductively. Let $+ = \textsc{dr}(\mathrm{id}_0^{1\to1}, 1^{3\to1})$ and $\cdot = \textsc{dr}(0^{1\to1}, \mathrm{id}_0^{3\to1})$. For inv_+, define

$$f = \textsc{dr}\big(1^{0\to1}, \cdot \circ \textsc{jx}\big(-1^{1\to1}, \cdot \circ \textsc{jx}(\mathrm{id}_0^{1\to1}, \mathrm{id}_0^{1\to1})\big) \circ \mathrm{id}_1^{2\to1}\big), \tag{10}$$

$$\mathrm{inv}_+ = f \circ \big(+ \circ \textsc{jx}(\mathrm{id}_0^{1\to1}, -1^{1\to1})\big), \tag{11}$$

or, more colloquially, $f t = 1 - \int_0^t (f\tau)^2 \cdot \mathrm{d}\tau$ and $\mathrm{inv}_+ t = f(t-1)$. Square root is defined similarly by

$$g t = 1 + \int_0^t \mathrm{inv}_+(2 \cdot g\tau) \cdot \mathrm{d}\tau, \qquad \mathrm{sqrt}_+ t = g(t-1). \tag{12}$$

Logarithm and exponentiation are derived analogously through suitable integral equations. For the trigonometric functions, let

$$\sin t = \int_0^t \cos\tau \cdot \mathrm{d}\tau, \qquad \cos t = 1 - \int_0^t \sin\tau \cdot \mathrm{d}\tau. \tag{13}$$

Formally this is to say

$$\mathrm{trig} = \textsc{dr}\big(\textsc{jx}(0^{0\to1}, 1^{0\to1}), \textsc{jx}\big(\mathrm{id}_2^{3\to1}, \big(\cdot \circ \textsc{jx}(-1^{3\to1}, \mathrm{id}_1^{3\to1})\big)\big)\big), \tag{14}$$

and then let $\sin = \mathrm{id}_0^{2\to1} \circ \mathrm{trig}$ and $\cos = \mathrm{id}_1^{2\to1} \circ \mathrm{trig}$. $\square$

Some authors say “$1/x$ is real primitive recursive” to mean that inv_+ is. It is not clear how such claims without specification of domain can be justified in general.

3.3 Properties preserved under differential recursion

The reader may have found the process unwieldy when we picked uniq_{K_v} out of K_v in Definition 3.5. This can be simplified if we restrict our attention to real primitive recursive functions, thanks to the following facts known as the *Existence and Uniqueness Theorems* for initial value problems.

Theorem 3.8. *Let f, g, v and K_v be as in Definition 3.5. If g is an analytic*[4] *function, then each function in K_v is a restriction of uniq_{K_v}.*

[4]This result is often stated with a weaker assumption that g be (locally) *Lipschitz continuous*.

THEOREM 3.9. *If f and g are analytic functions with open domain, then so is* $\mathrm{DR}(f, g)$.

The proofs are well known [Walter 1998]. Theorem 3.8, also implied by the proof of Theorem 3.10 below, says that a solution of (8) may diverge to infinity somewhere but can never "branch" as in Figure 1, provided g is smooth enough. We therefore could have dispensed with K' and simply used K_v instead in the definition of $uniq_{K_v}$, as long as f and g are real primitive recursive functions and hence analytic by Theorem 3.9.

In the following, we show that the real primitive recursive functions are computable in the sense of Computable Analysis. We follow the formulation by Weihrauch [2000] based on *representations* of real numbers, sets and functions by infinite strings. Formally, a representation γ of set A is a partial function from $\Sigma^{\mathbf{N}}$ onto A, where Σ is a finite alphabet (containing all symbols used below). When $\gamma p = x$, we say that the infinite string p is a γ-*name* of point x. Define the representation ρ of $\mathbf{R}$ as follows: $\rho p = x$ if and only if p lists (under some reasonable ways to encode and delimit rational numbers) all nonempty open intervals with rational endpoints that contain x. If sets $A_0, \ldots, A_{m-1}$ have representations $\alpha_0, \ldots, \alpha_{m-1}$, respectively, we can construct the representation $[\alpha_0, \ldots, \alpha_{m-1}]$ of $A_0 \times \cdots \times A_{m-1}$ by letting $[\alpha_0, \ldots, \alpha_{m-1}]p = (x_0, \ldots, x_{m-1})$ if and only if p is of form $\langle p_0, \ldots, p_{m-1}\rangle$ and $\alpha_0 p_0 = x_0, \ldots, \alpha_{m-1} p_{m-1} = x_{m-1}$, where $\langle \cdot, \ldots, \cdot \rangle$ is any reasonable tupling function on infinite strings. Write ρ^m for the representation $[\rho, \ldots, \rho]$ of $\mathbf{R}^m$.

Computation is regarded as conversion between such infinite strings by a Turing machine: a (partial) function $f : A \rightharpoonup B$, with A and B having representations α and β, is said to be (α, β)-*computed* by machine M if, given any α-name of any $x \in \operatorname{dom} f$ on the input tape, M outputs some β-name of $f x$ on the one-way output tape. Note that since names are infinite strings, M is not supposed to halt, but instead expected to write symbols out indefinitely. A set $X \subseteq A$ is α-*semi-decided* by M if, given $x \in A$ in the same manner, M halts when and only when $x \in X$.

We can also consider giving M access to another infinite string q as oracle in addition to the input name. It can be shown that all open subsets of $\mathbf{R}^m$ are ρ^m-semi-decided by some machine with some oracle, and that all continuous functions from $\mathbf{R}^m$ to $\mathbf{R}^n$ with open domain are (ρ^m, ρ^n)-computed by some machine with some oracle. This allows us to define representations of open sets and continuous functions as follows. Define[5] the representation $\theta_<$ of the set of open subsets of $\mathbf{R}^m$ by letting $\theta_<(M\#q) = X$ (with # a delimiter) if and only if M is (an encoding of) a machine that, given q as oracle, ρ^m-semi-decides X. Also define the representation $\delta^X_\to$ of the set of continuous functions from $\mathbf{R}^m$ to $\mathbf{R}^n$ with domain X by $\delta^X_\to(M\#q) = f$ if and only if M is a machine that, given q as oracle, (ρ^m, ρ^n)-computes f. These representations ρ, $\theta_<$ and $\delta^X_\to$ are equivalent to those used by Weihrauch [2000, Definitions 4.1.3, 5.1.15, 6.1.1]. Finally, let $F^{m \rightharpoonup n}$ denote the set of all continuous functions from $\mathbf{R}^m$ to $\mathbf{R}^n$ with open domain, and define its representation $\delta_{\preceq}$ by $\delta_{\preceq}(\langle p, q\rangle) = f$ if and only if $\theta_< p = \operatorname{dom} f$ and $\delta^{\theta_< p}_\to q = f$.

[5] We have this representation $\theta_<$ for each arity m, but following Weihrauch [2000, Definition 5.1.15] we will not indicate m explicitly. Similarly for $\delta^X_\to$ below.

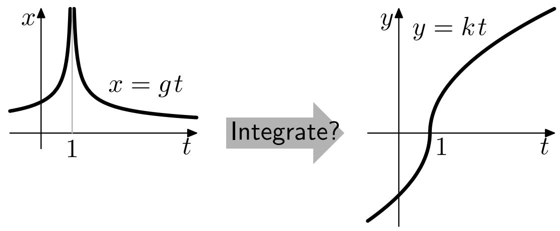

Fig. 2. Integrand with a singularity.

THEOREM 3.10. *Let* $m, n \in \mathbf{N}$. *There is a* $([\delta_{\trianglelefteq}^{\leq}, \delta_{\trianglelefteq}^{\leq}], \delta_{\trianglelefteq}^{\leq})$*-computable total function* $S : F^{m \rightharpoonup n} \times F^{m+1+n \rightharpoonup n} \rightharpoonup F^{m+1 \rightharpoonup n}$ *such that* $S(f, g) = \mathrm{DR}(f, g)$ *for all analytic functions* $f \in F^{m \rightharpoonup n}$ *and* $g \in F^{m+1+n \rightharpoonup n}$.

The proof is largely routine error estimation of Euler's method, see Appendix B. A slightly different statement appears in a recent work by Graça et al. [to appear]. They further show essentially that our representation $\theta_<$ approximating open sets "from inside" cannot be replaced by $\theta_>$ [Weihrauch 2000, Definition 5.1.15] that approximates "from outside."

COROLLARY 3.11. *Every real primitive recursive function has open domain, is analytic and has a computable* $\delta_{\trianglelefteq}^{\leq}$*-name.*

PROOF. This property is possessed by $0^{0 \to 1}$, $1^{0 \to 1}$ and $-1^{0 \to 1}$ and preserved under JX, CM and, by Theorems 3.9 and 3.10, under DR. □

A function f having a computable $\delta_{\trianglelefteq}^{\leq}$-name means, in Weihrauch's terminology, that it has a recursively enumerable open domain and is computable. In other words, there is a machine that, given any (ρ^m-name of) $x \in \mathbf{R}^m$ and (ρ-name of) positive $\varepsilon \in \mathbf{R}$, halts and outputs a rectangle of size ε with rational vertices containing $f x$ if $x \in \operatorname{dom} f$, and never halts otherwise.

3.4 Campagnolo's differential recursion

The clauses (a)–(c) of Definition 3.5 guarantee that equation (8) makes sense for all $t \in \operatorname{dom} k$. However, (c) could be slightly relaxed, since a small set of singularities in the integrand does not affect the integral. Define $\mathrm{DR_C}$ by replacing (c) with

(c′) $(v, \tau, k\tau) \in \operatorname{dom} g$ for any $\tau \in \operatorname{dom} k \setminus S$, where S is a countable set of isolated points.

This idea is due to Campagnolo [2001, Definition 2.4.2]. The choice between (c) and (c′) is somewhat analogous to the discussion regarding Principle 3.3. The issue is whether $g(v, \tau, k\tau)$, where $\tau \in [0, t]$, is a "subexpression" of the right-hand side of the equation (8). Without going into the philosophical discussion to ask which is "natural," we point out some differences this choice incurs.

Analyticity is no longer preserved, as the following example shows (Figure 2). Define $g : \mathbf{R} \rightharpoonup \mathbf{R}$ by $\operatorname{dom} g = \mathbf{R} \setminus \{1\}$ and

$$g t = \frac{1}{\sqrt{|t-1|}} = \mathrm{inv}_+\big(\mathrm{sqrt}_+\big(\mathrm{sqrt}_+(t-1)^2\big)\big). \tag{15}$$

It is real primitive recursive by Lemma 3.7. But $a = \mathrm{DR_C}(-2^{0\to 1}, g \circ \mathrm{id}_0^{2\to 1})$, where $-2^{0\to 1}$ is the constant function with value -2, is the total function

$$a\,t = \begin{cases} +2 \cdot \sqrt{t-1} & \text{if } t \geqslant 1, \\ -2 \cdot \sqrt{-t+1} & \text{if } t < 1, \end{cases} \tag{16}$$

which is not differentiable at 1. Note that $\mathrm{DR}(-2^{0\to 1}, g \circ \mathrm{id}_0^{2\to 1})$ is its restriction to $(-\infty, 1)$ and thus analytic. For a subtler example, recall the equation (12) for sqrt_+; with $\mathrm{DR_C}$, the same equation yields the square root function defined on $[0, \infty)$, rather than on $(0, \infty)$.

This breaks the assumption of Theorem 3.8 and thus gives rise to incomparable functions in K_v when, say, $f = 1^{0\to 1}$ and $g = a \circ \mathrm{id}_1^{2\to 1}$, with a from (16); that is, the equation $h t = 1 + \int_0^t a(h\tau) \cdot \mathrm{d}\tau$ has two solutions that disagree somewhere in the intersection of their domains.

Keeping the class analytic also matches the intention of Moore [1996, Definition 9] to make equation (8) equivalent to the differential equation

$$k\,0 = f\,v, \qquad \mathrm{D}^1 k\,t = g(v, t, k\,t), \tag{17}$$

which would not make sense for non-differentiable h.

4. FUNCTIONS THAT ARE NOT DIFFERENTIALLY ALGEBRAIC

Claim 1.1 would not make sense if we adopted $\mathrm{DR_C}$ in defining real primitive recursive functions, because there would then arise non-analytic functions, as we noted above. We now show that, even under our restrictive definition with the analyticity-preserving DR, the claim fails.

Define functions g, $k : \mathbf{R}^2 \rightharpoonup \mathbf{R}$ by $\mathrm{dom}\, g = \mathrm{dom}\, k = (0, \infty) \times (-1, \infty)$ and

$$g(x,t) = \mathrm{e}^{(x-1)\cdot \ln(t+1) - (t+1)}, \qquad k(x,t) = \int_0^t g(x,\tau) \cdot \mathrm{d}\tau. \tag{18}$$

Note that g and k are real primitive recursive by Lemma 3.7, and that g is differentially algebraic by Lemma 2.3 and Theorem 3.4. Euler's *gamma function* $\mathit{\Gamma} : \mathbf{R} \rightharpoonup \mathbf{R}$ is defined by $\mathrm{dom}\, \mathit{\Gamma} = (0, \infty)$ and

$$\mathit{\Gamma} x = \lim_{t\to\infty} \check{\mathit{\Gamma}}(x,t), \qquad \text{where } \check{\mathit{\Gamma}}(x,t) = k(x, t-1) - k\left(x, \frac{1}{t} - 1\right). \tag{19}$$

It can be verified [Artin 1964, Chapter 2] that $\mathit{\Gamma}$ converges on $(0, \infty)$ and that

$$\mathrm{D}^n \mathit{\Gamma} x = \lim_{t\to\infty} \mathrm{D}^{(n,0)} \check{\mathit{\Gamma}}(x,t), \qquad n \in \mathbf{N},\ x \in (0,\infty). \tag{20}$$

Hölder showed that $\mathit{\Gamma}$ is not differentially algebraic [Hölder 1886; Moore 1896].

Lemma 4.1. *The function k defined above is not differentially algebraic.*

Proof. If it were, then so would be $\check{\mathit{\Gamma}}$ by Lemma 2.3 and Theorem 3.4, giving a nonzero polynomial P satisfying

$$P\left(\check{\mathit{\Gamma}}(x,t), \mathrm{D}^{(1,0)} \check{\mathit{\Gamma}}(x,t), \ldots, \mathrm{D}^{(\mathrm{arity}\, P - 1, 0)} \check{\mathit{\Gamma}}(x,t)\right) = 0, \qquad (x,t) \in (0,\infty)^2. \tag{21}$$

Note that we used (i) of Theorem 2.1 to take P independently of t. We take the limit of (21) as $t \to \infty$, which by (20) yields $P(\mathit{\Gamma} x, \mathrm{D}^1 \mathit{\Gamma} x, \ldots, \mathrm{D}^{\mathrm{arity}\, P - 1} \mathit{\Gamma} x) = 0$, contradicting Hölder. □

This already disproves Claim 1.1, but let us take a step further to address one of the questions asked by Rubel [1992, Problem 31]:

> Given a "nice" initial-value problem for a system of algebraic differential equations in the dependent variables $y_1, \ldots, y_n$, must $y_1(x_0)$ be differentially algebraic as a function of the *initial conditions*, for each x_0? We won't say more about what "nice" means except that the problem should have a unique solution for each initial condition in a suitable open set.

Katriel [2003] already gives a negative answer, but we will present another counterexample. Consider the system of equations

$$\mathrm{D}^1 y_1 t = y_2 t, \quad \mathrm{D}^1 y_2 t = y_3 t \cdot y_2 t, \quad \mathrm{D}^1 y_3 t = -y_4 t \cdot (y_3 t + 1)^2, \quad \mathrm{D}^1 y_4 t = 0 \tag{22}$$

for $t \in (-1, \infty)$. By Theorem 3.8, it seems safe to say that this system satisfies the "niceness" required by Rubel. We claim that $y_1 t$ is not differentially algebraic as a function of the initial conditions $y_1 0$, $y_2 0$, $y_3 0$, $y_4 0$ and t. For suppose that it were. Then by Lemma 2.3 and Theorem 3.4, so would be $y_1 t$ as a function of $x \in (0, \infty)$ and t if we set the initial condition to be $(y_1 0, y_2 0, y_3 0, y_4 0) = (0, 1/\mathrm{e}, x - 2, 1/(x-1))$. But it is easy to see that this initial condition yields the solution

$$y_4 t = \frac{1}{x-1}, \qquad y_3 t = \frac{x-1}{t+1} - 1, \qquad y_2 t = g(x,t), \qquad y_1 t = k(x,t), \tag{23}$$

where g and k are from (18). This contradicts Lemma 4.1.

By (iii) of Theorem 2.1, this answers Rubel's question negatively, showing that for some t_0, the unary function $k(\cdot, t_0)$ is not differentially algebraic. In fact, the proof of Theorem 2.1 shows this for *most* t_0, in the sense that the set of t_0 for which $k(\cdot, t_0)$ is differentially algebraic has empty interior. We do not know, however, whether this small set contains all numbers that can be written as $f()$ for a real primitive recursive function $f : \mathbf{R}^0 \rightharpoonup \mathbf{R}$. A negative answer to this would provide a unary counterexample to Claim 1.1. We have been unable to find such an example.

5. OTHER CLASSES AND RELATED WORKS

This section discusses some other operators introduced by Moore and other authors.

5.1 Zero-finding and real recursive functions

For a function $f : \mathbf{R}^{m+1} \rightharpoonup \mathbf{R}$, Moore defines $\textsc{zf}\, f : \mathbf{R}^m \rightharpoonup \mathbf{R}$ by

$$\textsc{zf}\, f\, v = \begin{cases} t^+ = \inf \{\, t \geq 0 \mid f(v,t) = 0 \,\} & \text{if } t^+ < -t^-, \\ t^- = \sup \{\, t \leq 0 \mid f(v,t) = 0 \,\} & \text{otherwise.} \end{cases} \tag{24}$$

The class of *real recursive* functions[6] is defined to be the smallest class containing real primitive recursive functions and closed under JX, CM, DR and ZF. Moore states the definition of ZF in a way that leaves ambiguous whether (24) has a value when, say, $\operatorname{dom} f = \mathbf{R}^m \times [1, \infty)$ and $f(v,t) = 2 - t$ for all $t \geq 1$. Should it have the

[6]This "recursiveness" of Moore's has little to do with the same word often used in Computable Analysis. As we see in Appendix C, Moore's real recursive functions can even be discontinuous.

value 2, or be left undefined because "the zero-searching program starting from the origin gets stuck"?

It turns out that, whichever definition we choose, we can keep Moore's claim on iteration [Moore 1996, Proposition 11] in the following modified form. Since the original proof again forgets partial functions, we sketch a new proof in Appendix C.

LEMMA 5.1. *If* $f : \mathbf{R}^m \rightharpoonup \mathbf{R}^m$ *is real recursive, there is a real recursive function* $g : \mathbf{R}^{m+1} \rightharpoonup \mathbf{R}^m$ *that extends the function* g' *defined by* $\operatorname{dom} g' = \{\, (v,k) \in \mathbf{R}^m \times (\mathbf{N} \setminus \{0\}) \mid v \in \operatorname{dom} f^k \,\}$ *and* $g'(v,k) = f^k v$ *for all* $(v,k) \in \operatorname{dom} g'$*, where* $f^k = \underbrace{f \circ \cdots \circ f}_{k}$.

We have to note, however, that the class of real recursive functions is probably not well-behaved, since, with ZF producing non-smooth functions, the class no longer enjoys Theorem 3.8. This makes Claim 1.2 less philosophically appealing, although it could be proved by using Lemma 5.1 to simulate Turing machines as Moore did.

5.2 Related works and open issues

We have seen that most problems in Moore's original work were caused by failure to deal with partial functions properly. Some authors avoid this trouble by studying only operators preserving totality, so that partial functions never come into discussion. Campagnolo and Moore [2001] take this path by considering *linear differential recursion* in place of DR. For classes defined by this operator (and sometimes additional operators), some relations to digital computation are known [Campagnolo 2001; Bournez and Hainry 2006; Campagnolo and Ojakian 2008], see also the survey [Bournez and Campagnolo 2008].

We have refuted a claimed inclusion of Moore's class in the differentially algebraic (or GPAC generable) functions. Stating a converse direction requires some care, because there are uncountably many differentially algebraic functions. Moore [1996, Proposition 9] formulates this direction by restricting the initial conditions in the GPAC. Pour-El [1974, Section 4] shows that GPAC generable functions are "essentially" computable in a traditional sense. Some other authors [Miller 1970; Pour-El and Richards 1979; Ko 1983; 1992; Edalat and Pattinson 2004] also study equations of form (7) and their variants in the context of Computable Analysis.

Claim 1.1 has been one of the main rationales for calling variants of Moore's classes a model of analog computation. It is therefore an important challenge to look for a subclass of our real primitive recursive functions, preferably with an equally simple definition, that has a close relationship to the differentially algebraic functions. Another direction would be to reformulate further the rest of Moore's work, as well as other authors' works that also suffer from the same inaccuracy. For example, it may be interesting to work out Mycka and Costa's class arising from the operator of taking limits [Mycka and Costa 2004; Loff et al. 2007].

In this paper, we stayed within mathematics and chose not to engage in any physical interpretation of analog models. For Shannon's and Moore's models, parts of the original papers present such discussion. A more philosophical question of whether analog devices can be more powerful than digital computers has been addressed by some authors [Vergis et al. 1986; Penrose 1989; Yao 2003].

A. OLD RESULTS

We list some known theorems that we used in Section 2. Let $J \subseteq \mathbf{R}$ be an open interval. It is well known that functions $u_0, \ldots, u_{k-1} \in \mathrm{C}^\omega[J]$ are linearly dependent if and only if the determinant $\left|(\mathrm{D}^i u_j)_{i,j=0,\ldots,k-1}\right|$, called their *Wronskian*, is zero. Using this fact, Ritt and Gourin [1927] showed (iii$_\mathbf{R}$) ⇒ (iii) of Theorem 2.1.

THEOREM A.1. *Let $J \subseteq \mathbf{R}$ be an open interval and let $f \in \mathrm{C}^\omega[J]$. If*

$$P(f, \mathrm{D}^1 f, \mathrm{D}^2 f, \ldots, \mathrm{D}^{\operatorname{arity} P - 1} f) = 0 \tag{25}$$

for some $\mathbf{R}$-coefficient nonzero polynomial P, then we have in fact (25) *for some $\mathbf{Z}$-coefficient nonzero polynomial P.*

PROOF. By the assumption, there is a finite set $K \subseteq \mathbf{N}^l$, where $l = \operatorname{arity} P$, such that the functions

$$f^{\nu_0} \cdot (\mathrm{D}f)^{\nu_1} \cdots (\mathrm{D}^{l-1} f)^{\nu_{l-1}}, \qquad (\nu_0, \ldots, \nu_{l-1}) \in K, \tag{26}$$

are linearly dependent. The Wronskian of (26) thus vanishes, which is a $\mathbf{Z}$-coefficient nonzero polynomial in f, $\mathrm{D}^1 f$, …, $\mathrm{D}^{l+|K|-1} f$. □

The following *Baire Category Theorem* was used in another part of Theorem 2.1.

THEOREM A.2. *Let J be a subset of $\mathbf{R}^m$. The union of countably many closed subsets of J with empty interior has empty interior.*

PROOF. Let J_0, J_1, … be closed subsets of J with empty interior, and U be any nonempty open subset of J. We will show that $U \setminus \bigcup_{P \in \mathbf{N}} J_P$ is nonempty. For each $P \in \mathbf{N}$, we take $x_P \in \mathbf{R}^m$ and $\varepsilon_P \in \mathbf{R}$ as follows. Write $\mathrm{B}(x, \varepsilon)$ for the open set of points in J whose distance from x is less than ε. Let $x_0 \in U$ and $\varepsilon_0 \in (0, 1)$ be such that $\mathrm{B}(x_0, \varepsilon_0) \subseteq U$. For each $P \in \mathbf{N}$, let $x_{P+1} \in U$ and $\varepsilon_{P+1} \in (0, 2^{-P-1})$ be such that $\mathrm{B}(x_{P+1}, \varepsilon_{P+1}) \subseteq \mathrm{B}(x_P, \varepsilon_P) \setminus J_P$. This is possible because $\mathrm{B}(x_P, \varepsilon_P) \setminus J_P$ is open and nonempty, since J_P is closed and has empty interior. As P tends to infinity, x_P converges to a point in $U \setminus \bigcup_{P \in \mathbf{N}} J_P$. □

One direction of Lemma 2.4 used the following *Transcendence Degree Theorem.*

THEOREM A.3. *Let F be a subfield of a field E and D be a subset of E. If $D \subseteq \overline{F(B)}$ for some finite set $B \subseteq E$, then $D \subseteq \overline{F(C)}$ for some finite set $C \subseteq D$.*

PROOF. By assumption, each $d \in D$ satisfies $P_d d = 0$ for some $F(B)$-coefficient nonzero polynomial P_d. We may assume that the coefficients of P_d are F-coefficient polynomials in the elements of B. We further assume, since otherwise we are done, that there are $d_0 \in D$ and $b \in B \setminus D$ such that at least one of the coefficients of P_{d_0} has positive degree in b. In this case, we can write $Qb = P_{d_0} d_0$ for some $F(B')$-coefficient nonzero polynomial Q, where $B' = (B \setminus \{b\}) \cup \{d_0\}$.

For any $d \in D$ and $t \in \mathbf{N}$, we can use $P_d d = 0$ and then $Qb = 0$ to write

$$\begin{aligned} d^t &= \sum_{i=0}^{\deg P_d - 1} \beta_i \cdot d^i && \text{for some } \beta_i \in F(B) \\ &= \sum_{i=0}^{\deg P_d - 1} \sum_{j=0}^{\deg Q - 1} \gamma_{i,j} \cdot b^i \cdot d^i && \text{for some } \gamma_{i,j} \in F(B'). \end{aligned} \tag{27}$$

This means that d, d^2, $\underline{d^3, \ldots}$ all belong to a vector space of finite dimension over $F(B')$, and hence $d \in \overline{F(B')}$. We have thus found a set B' with $D \subseteq \overline{F(B')}$ such that $B' \setminus D$ is strictly smaller than $B \setminus D$. Repeat. $\square$

B. EFFECTIVE SOLUTION

This section proves Theorem 3.10, the computability of differential recursion on analytic inputs. The proof will use the UTM and SMN Theorems without notice. In particular, we have the following lemmas that are easy to verify. Let $\nu_{\mathbf{N}}$ be the representation of $\mathbf{N}$ such that a $\nu_{\mathbf{N}}$-name of a nonnegative integer is its binary notation followed by a special termination symbol and then any infinite string.

LEMMA B.1. *Let α be a representation of set A and let $s : A \rightharpoonup A$ be an (α, α)-computable function with α-semi-decidable domain. Then the function $s^* : \mathbf{N} \times A \rightharpoonup \mathbf{N}$ defined by*

$$s^*(N, x) = s^N x = (\underbrace{s \circ s \circ \cdots \circ s}_{N})x \tag{28}$$

is $([\nu_{\mathbf{N}}, \alpha], \alpha)$-computable and has $[\nu_{\mathbf{N}}, \alpha]$-semi-decidable domain.

LEMMA B.2. *Let α be a representation of set A. A function T from A to the set of open subsets of $\mathbf{R}^m$ is $(\alpha, \theta_<)$-computable if and only if the set $\{ (x, v) \mid v \in Tx \}$ is $[\alpha, \rho^m]$-semi-decidable. A total function $S : A \rightharpoonup F^{m \rightharpoonup n}$ is $(\alpha, \delta_{\trianglelefteq}^{\leq})$-computable if and only if $\operatorname{dom} \circ S$ is $(\alpha, \theta_<)$-computable and the function $\overline{S}$ defined by $\operatorname{dom} \overline{S} = \bigcup_{x \in A} \{x\} \times \operatorname{dom}(Sx)$ and $\overline{S}(x, v) = Sxv$ is $([\alpha, \rho^m], \rho^n)$-computable.*

Since we will always represent $\mathbf{N}$, $\mathbf{R}$ and $F^{m \rightharpoonup n}$ by $\nu_{\mathbf{N}}$, ρ and $\delta_{\trianglelefteq}^{\leq}$, we will omit the list of representations prefixed to computability and semi-decidability.

PROOF OF THEOREM 3.10. For simplicity, we assume $n = 1$. The proof can be straightforwardly generalized by reading inequalities, intervals, maxima and minima below in a "componentwise" way. We will construct two functions A^-, $A^+ : F^{m \rightharpoonup 1} \times F^{m+1+1 \rightharpoonup 1} \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{N} \times \mathbf{N} \rightharpoonup \mathbf{R}$ with common domain such that

(A) $A^\pm$ are computable and $\operatorname{dom} A^\pm$ is semi-decidable;

(B) if $(v, \hat{t}) \in \operatorname{dom} h$, where $h = \textsc{dr}(f, g)$, then for each $\varepsilon > 0$ there is $(M, N) \in \mathbf{N}^2$ such that $(f, g, v, \hat{t}, M, N) \in \operatorname{dom} A^\pm$ and $A^-(f, g, v, \hat{t}, M, N) \leq h(v, \hat{t}) \leq A^+(f, g, v, \hat{t}, M, N) < A^-(f, g, v, \hat{t}, M, N) + \varepsilon$;

(C) otherwise, $(f, g, v, \hat{t}, M, N) \notin \operatorname{dom} A^\pm$ for all $(M, N) \in \mathbf{N}^2$.

Once this is done, the theorem follows by Lemma B.2.

Our presentation below will assume $\hat{t} > 0$ for simplicity; it should be obvious what to do to make it work for all inputs. Define *step* : $F^{m \rightharpoonup 1} \times F^{m+1+1 \rightharpoonup 1} \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \mathbf{R} \times \mathbf{R} \rightharpoonup F^{m \rightharpoonup 1} \times F^{m+1+1 \rightharpoonup 1} \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \mathbf{R} \times \mathbf{R}$ as follows: $(f, g, v, \hat{t}, M, N, i, k^-, k^+)$ belongs to dom *step* if and only if $(v, t, y) \in \operatorname{dom} g$ and $|g(v, t, y)| < M$ for all $(t, y) \in T \times Y$, where $T = [i \cdot \hat{t}/N, (i+1) \cdot \hat{t}/N]$ and $Y = [k^- - M \cdot \hat{t}/N, k^+ + M \cdot \hat{t}/N]$; it is then mapped to

$$\left(f, g, v, \hat{t}, M, N, i+1, k^- + \int_T \min_{y \in Y} g(v, t, y) \cdot \mathrm{d}t, k^+ + \int_T \max_{y \in Y} g(v, t, y) \cdot \mathrm{d}t \right). \tag{29}$$

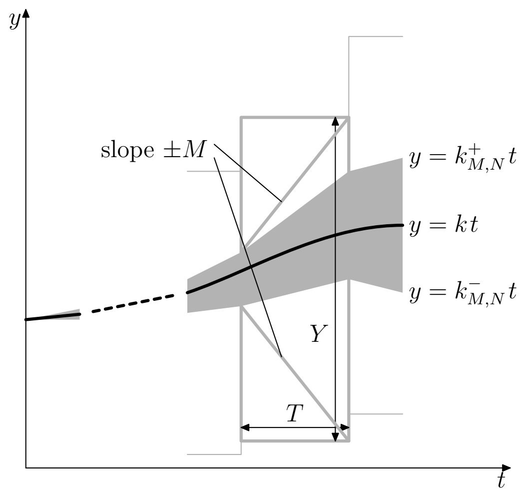

Fig. 3. The slopes of $k^{\pm}_{M,N}$ on T are the maximum and minimum of g on $T \times Y$, and thus $k^{\pm}_{M,N}$ give an upper and a lower bound for k.

Because maximization and integration are computable [Weihrauch 2000, Corollary 6.2.5 and Theorem 6.4.1], *step* is computable and has semi-decidable domain. Define piecewise linear functions $k^-_{M,N}$, $k^+_{M,N} : \mathbf{R} \rightharpoonup \mathbf{R}$ (depending on f, g, v, $\hat{t}$, M and N) with common domain as follows: $\operatorname{dom} k^{\pm}_{M,N}$ is empty if $v \notin \operatorname{dom} f$, and otherwise equals $[0, j \cdot \hat{t}/N]$ for the last $j \le N$ with $(f, g, v, \hat{t}, M, N, 0, f\,v, f\,v) \in \operatorname{dom} \mathit{step}^j$; for each $i \le j$ then, $k^-_{M,N}(i \cdot \hat{t}/N)$ and $k^+_{M,N}(i \cdot \hat{t}/N)$ are the last two components of $\mathit{step}^i(f, g, v, \hat{t}, M, N, 0, f\,v, f\,v)$; interpolate $k^{\pm}_{M,N}$ linearly at other points (Figure 3). Let $A^{\pm}(f, g, v, \hat{t}, M, N) = k^{\pm}_{M,N}\,\hat{t}$. We have (A) by Lemma B.1.

Assume the hypothesis of (B) and let $\varepsilon > 0$. By the definition of DR, we have a function $k : \mathbf{R} \rightharpoonup \mathbf{R}$ satisfying the four clauses in Definition 3.5 and $\hat{t} \in \operatorname{dom} k$. Since the graph of k is included in the open set $\{\, (t, y) \mid (v, t, y) \in \operatorname{dom} g \,\}$ by (c), so is the compact strip $W_\delta = \bigcup_{t \in [0,\hat{t}]} \{t\} \times [k\,t - 2 \cdot \delta, k\,t + 2 \cdot \delta]$ for some $\delta > 0$. We may assume that $\delta < \varepsilon$. Let M be so large that

$$M > |g(v, t, y)|, \tag{30}$$

$$M > |\mathrm{D}^{(0,\dots,0,1)} g(v, t, y)| \tag{31}$$

for each $(t, y) \in W_\delta$, and then let N be so large that

$$2 \cdot M \cdot \frac{\hat{t}}{N} < \delta \cdot \mathrm{e}^{-2 \cdot M \cdot \hat{t}}. \tag{32}$$

We show by induction on $i = 0, \dots, N$ that

(D_i) $k^+_i = k^+_{M,N}(i \cdot \hat{t}/N)$ and $k^-_i = k^-_{M,N}(i \cdot \hat{t}/N)$ are defined,

(P_i) $k^+_i - k^-_i < \delta \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N}$, and

(Q_i) $k^-_i \le k(i \cdot \hat{t}/N) \le k^+_i$.

Since $v \in \operatorname{dom} f$ by (b) of Definition 3.5, we have (D_0), (P_0) and (Q_0). Assume (D_i), (P_i) and (Q_i). By (Q_i) and (30), we have $k\,t \in Y$ for all $t \in T$, where

$T = [i \cdot \hat{t}/N, (i+1) \cdot \hat{t}/N]$ and $Y = [k_i^- - M \cdot \hat{t}/N, k_i^+ + M \cdot \hat{t}/N]$. Hence $T \times Y \subseteq W_\delta$, because the length of Y is $< 2 \cdot \delta$ by (P_i) and (32). This and (30) imply $(f, g, v, \hat{t}, M, N, i, k_i^-, k_i^+) \in \operatorname{dom} \mathit{step}$ and thus (D_{i+1}). The definition of *step* and the above fact that $kt \in Y$ for all $t \in T$ show that (Q_i) implies (Q_{i+1}). For (P_{i+1}), calculate

$$\begin{aligned}
k_{i+1}^+ - k_{i+1}^- &= (k_i^+ - k_i^-) + \int_T \bigl(\max_{y \in Y} g(v,t,y) - \min_{y \in Y} g(v,t,y)\bigr) \cdot \mathrm{d}t \\
&\le (k_i^+ - k_i^-) + \int_T \Bigl(2 \cdot M \cdot \frac{\hat{t}}{N} + (k_i^+ - k_i^-)\Bigr) \cdot M \cdot \mathrm{d}t \\
&< \delta \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N} + \frac{\hat{t}}{N} \cdot \Bigl(2 \cdot M \cdot \frac{\hat{t}}{N} + \delta \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N}\Bigr) \cdot M \\
&\le \delta \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N} + \frac{\hat{t}}{N} \cdot \bigl(\delta \cdot \mathrm{e}^{-2 \cdot M \cdot \hat{t}} + \delta \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N}\bigr) \cdot M \\
&\le \delta \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N} + \delta \cdot 2 \cdot M \cdot \frac{\hat{t}}{N} \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N} \\
&\le \delta \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N} + \delta \cdot \bigl(\mathrm{e}^{-2 \cdot M \cdot (N-i-1) \cdot \hat{t}/N} - \mathrm{e}^{-2 \cdot M \cdot (N-i) \cdot \hat{t}/N}\bigr) \\
&= \delta \cdot \mathrm{e}^{-2 \cdot M \cdot (N-i-1) \cdot \hat{t}/N},
\end{aligned} \tag{33}$$

where the first three inequalities are by (31), (P_i) and (32), respectively. The conclusions of (B) follow from (D_N), (P_N) and (Q_N).

For (C), suppose $(f, g, v, \hat{t}, M, N) \in \operatorname{dom} A^\pm$ for some (M, N). This means that $\operatorname{dom} k_{M,N}^\pm = [0, \hat{t}]$. It is not hard to see then that $\operatorname{dom} k_{M,2 \cdot N}^\pm = [0, \hat{t}]$ and that

$$k_{M,N}^- t \le k_{M,2 \cdot N}^- t \le k_{M,2 \cdot N}^+ t \le k_{M,N}^+ t, \qquad t \in [0, \hat{t}]. \tag{34}$$

Moreover, since (32) holds with N doubled and δ halved, $k_{M,2 \cdot N}^+ \hat{t} - k_{M,2 \cdot N}^- \hat{t} < \delta/2$ by an induction argument similar to (33). By (34) and this, $k_{M,2^c \cdot N}^\pm$ converges to a function k as $c \to \infty$. This k satisfies (8) for all $t \in [0, \hat{t}]$, because (8) holds with k and $=$ replaced respectively by $k_{M,2^c \cdot N}^+$ and $\ge$ and by $k_{M,2^c \cdot N}^-$ and $\le$. □

The analyticity assumption was used only to give the derivative and its bound M in (31). Though M is not read by the machine, it is necessary for the proof of its correctness. Dropping this assumption may cause Theorems 3.8 and 3.9 to fail, and the (non-unique) solutions to be all non-computable [Pour-El and Richards 1979].

The proof above can be viewed as an effective version of Theorem 3.8. Another standard proof uses the fixed point theorem for contractive mappings [Walter 1998, Chapter 5]. We did not take this approach here, because it would have to involve more machinery of Computable Analysis to describe the function space, and also because a straightforward effectivization of such proofs seems to require that the bound M or some similar amount be referenced by the machine.

C. ITERATION

As we noted, the definition (24) of ZF is ambiguous, as its subexpression $f(v, t)$ may be undefined for some (v, t). So when is ZF $f\,v$ defined? Possible answers include:

(1) When t^+ and t^- are defined.

(2) When at least either t^+ or t^- is defined; the condition $t^+ < -t^-$ will be used only when both are defined.

And when is t^+ (resp. t^-) defined? Possible answers include:

(i) When there is $t \geq 0$ (resp. ≤ 0) such that $f(v,t) = 0$ and $(v,\tau) \in \operatorname{dom} f$ for all $\tau \in \mathbf{R}$.

(ii) When there is $t \geq 0$ (resp. ≤ 0) such that $f(v,t) = 0$ and $(v,\tau) \in \operatorname{dom} f$ for all $\tau \in [-t,t]$ (resp. $[t,-t]$).

(iii) When there is $t \geq 0$ (resp. ≤ 0) such that $f(v,t) = 0$ and $(v,\tau) \in \operatorname{dom} f$ for all $\tau \in [0,t]$ (resp. $[t,0]$).

(iv) When there is $t \geq 0$ (resp. ≤ 0) such that $f(v,t) = 0$.

For (i), (ii) and (iii), we may also consider adding the phrase "except for some countably many isolated τ," just like (c′) in Section 3.4.

Moore's informal explanation by a programming language [Moore 1996, Section 7] suggests (2) and (ii). However, without discussing which is the "right" definition of ZF, we show that, whichever we choose, Lemma 5.1 holds. The following construction is consistent with any of the above 2×7 possible definitions.

PROOF OF LEMMA 5.1. Denote ZF $f\,v$ by $\mu t.\, f(v,t)$. Let

$$\text{zero?}\, x = \mu y.\, (x^2 + y^2) \cdot (1 - y), \tag{35}$$

$$\text{integer?}\, x = \text{zero?}\big(\sin(\pi \cdot x)\big), \tag{36}$$

where $\pi = 4 \cdot \operatorname{Arctan} 1$, with Arctan defined by a suitable integral equation. Let

$$\text{round}\, x = x - \mu r.\, \text{integer?}(x - r), \tag{37}$$

so that (37) is the unique integer in $(x - 1/2, x + 1/2]$. We get $\overline{\text{inv}}$ of (5) by

$$\overline{\text{inv}}\, x = \mu t.\, x \cdot (x \cdot t - 1). \tag{38}$$

The above four functions are total. Let

$$\text{digit}(x,b,i) = \text{round}\left(\frac{x}{b^i} - \frac{1}{2}\right) - b \cdot \text{round}\left(\frac{x}{b^{i+1}} - \frac{1}{2}\right) \tag{39}$$

for $b > 0$, where $b^i = \exp(i \cdot \ln b)$. When $b > 1$ and i are integers, $\text{digit}(x,b,i)$ is the digit in b^i's place when x is written in base-b notation. Define

$$c\,t = \text{digit}(t,2,-1), \qquad z\,t = 0 + \int_0^t (2 - 4 \cdot c\,\tau) \cdot \mathrm{d}\tau, \tag{40}$$

$$g(v,t) = v + \int_0^t 2 \cdot (1 - c\,\tau) \cdot \Big(f\big(h(v,\tau) - c\,\tau \cdot (h(v,\tau) - v)\big) - h(v,\tau)\Big) \cdot \mathrm{d}\tau, \tag{41}$$

$$h(v,t) = v + \int_0^t 2 \cdot c\,\tau \cdot \big(h(v,\tau) - g(v,\tau)\big) \cdot \overline{\text{inv}}(z\,\tau) \cdot \mathrm{d}\tau \tag{42}$$

(Figure 4), where the last two equations are a mutual definition similar to (13). We have $f^k v = g(v, k - 1/2)$ for $k \in \mathbf{N} \setminus \{0\}$. □

Note that $c\,\tau \cdot (h(v,\tau) - v)$ in (41) cannot be dropped, because of Principle 3.3 for CM.

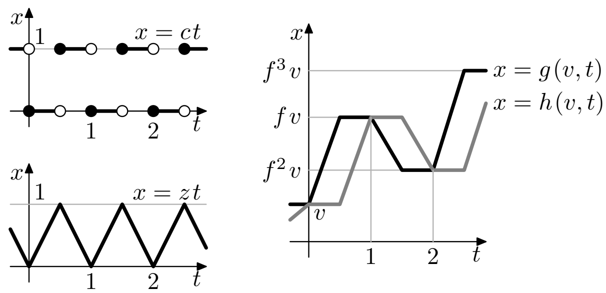

Fig. 4. Simulating iteration $f^k v$ by equations (41) and (42).

ACKNOWLEDGMENT

The author is grateful to Masami Hagiya and Mariko Yasugi for the discussion that led to this work; to Stephen A. Cook and José Félix Costa for their comments on the manuscript that helped improve the presentation substantially; and to the anonymous referees for providing further advice and informing him of important recent works [Katriel 2003; Bournez and Campagnolo 2008; Graça et al. to appear].

REFERENCES

Artin, E. 1964. *The Gamma Function.* Athena Series: Selected Topics in Mathematics. Holt, Rinehart and Winston, New York. Translated by Michael Butler from *Einführung in die Theorie der Gamma-Funktion*, 1931.

Blum, L., Cucker, F., Shub, M., and Smale, S. 1997. *Complexity and Real Computation.* Springer-Verlag.

Bournez, O. and Campagnolo, M. L. 2008. A survey on continuous time computations. In *New Computational Paradigms.* Springer New York, 383–423.

Bournez, O. and Hainry, E. 2006. Recursive analysis characterized as a class of real recursive functions. *Fundamenta Informaticae 74,* 4, 409–433.

Bush, V. 1931. The Differential Analyzer: A new machine for solving differential equations. *Journal of the Franklin Institute 212*, 447–488.

Campagnolo, M. L. 2001. Computational complexity of real valued recursive functions and analog circuits. Ph.D. thesis, Instituto Superior Técnico.

Campagnolo, M. L. and Moore, C. 2001. Upper and lower bounds on continuous-time computation. In *Second International Conference on Unconventional Models of Computation.* 135–153.

Campagnolo, M. L., Moore, C., and Costa, J. F. 2000. Iteration, inequalities, and differentiability in analog computers. *Journal of Complexity 16,* 4, 642–660.

Campagnolo, M. L. and Ojakian, K. 2008. The elementary computable functions over the real numbers: Applying two new techniques. *Archive for Mathematical Logic 46,* 7–8, 593–627.

Edalat, A. and Pattinson, D. 2004. A domain theoretic account of Picard's theorem. In *Proceedings of the 31st International Colloquium on Automata, Languages and Programming.* Lecture Notes in Computer Science, vol. 3142. 494–505.

Graça, D. 2002. The General Purpose Analog Computer and recursive functions over the reals. M.S. thesis, Instituto Superior Técnico.

Graça, D. 2004. Some recent developments on Shannon's General Purpose Analog Computer. *Mathematical Logic Quarterly 50*, 473–485.

Graça, D. S., Zhong, N., and Buescu, J. Computability, noncomputability and undecidability of maximal intervals of IVPs. To appear.

Grzegorczyk, A. 1955. Computable functionals. *Fundamenta Mathematicae 42*, 168–202.

Hölder, O. L. 1886. Ueber die Eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen. *Mathematische Annalen 28*, 1–13.

Katriel, G. 2003. Solution to Rubel's question about differentially algebraic dependence on initial values. *Illinois Journal of Mathematics 47,* 4, 1261–1272.

Kawamura, A. 2005. Type-2 computability and Moore's recursive functions. In *Proceedings of the 6th International Workshop on Computability and Complexity in Analysis*. Electronic Notes in Theoretical Computer Science, vol. 120. 83–95.

Ko, K.-I. 1983. On the computational complexity of ordinary differential equations. *Information and Control 58*, 157–194.

Ko, K.-I. 1992. On the computational complexity of integral equations. *Annals of Pure and Applied Logic 58,* 3, 201–228.

Krantz, S. G. and Parks, H. R. 2002. *A Primer of Real Analytic Functions*, Second ed. Birkhäuser Advanced Texts. Birkhäuser Boston.

Lipshitz, L. and Rubel, L. A. 1987. A differentially algebraic replacement theorem, and analog computability. *Proceedings of the American Mathematical Society 99,* 2, 367–372.

Loff, B., Costa, J. F., and Mycka, J. 2007. Computability on reals, infinite limits and differential equations. *Applied Mathematics and Computation 191,* 2, 353–371.

Miller, W. 1970. Recursive function theory and numerical analysis. *Journal of Computer and System Sciences 4*, 465–472.

Moore, C. 1996. Recursion theory on the reals and continuous-time computation. *Theoretical Computer Science 162*, 23–44.

Moore, E. H. 1896. Concerning transcendentally transcendental functions. *Mathematische Annalen 48,* 1-2, 49–74.

Mycka, J. and Costa, J. F. 2004. Real recursive functions and their hierarchy. *Journal of Complexity 20,* 6, 835–857.

Orponen, P. 1997. A survey of continuous-time computation theory. In *Advances in Algorithms, Languages, and Complexity*. Kluwer Academic Publishers, 209–224.

Ostrowski, A. 1920. Über Dirichletsche Reihen und algebraische Differentialgleichungen. *Mathematische Zeitschrift 8*, 241–298.

Penrose, R. 1989. *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics*. Oxford University Press, Inc., New York, NY, USA.

Pour-El, M. B. 1974. Abstract computability and its relation to the General Purpose Analog Computer (some connections between logic, differential equations and analog computers). *Transactions of the American Mathematical Society 199*, 1–28.

Pour-El, M. B. and Richards, I. 1979. A computable ordinary differential equation which possesses no computable solution. *Annals of Mathematical Logic 17,* 1–2, 61–90.

Ritt, J. F. and Gourin, E. 1927. An assemblage-theoretic proof of the existence of transcendentally transcendental functions. *Bulletin of the American Mathematical Society 33*, 182–184.

Rubel, L. A. 1992. Some research problems about algebraic differential equations II. *Illinois Journal of Mathematics 36,* 4, 659–680.

Shannon, C. E. 1941. Mathematical theory of the Differential Analyzer. *Journal of Mathematics and Physics 20,* 4, 337–354.

Vergis, A., Steiglitz, K., and Dickinson, B. 1986. The complexity of analog computation. *Mathematics and Computers in Simulation 28,* 2, 91–113.

Walter, W. 1998. *Ordinary differential equations*. Graduate Texts in Mathematics, Readings in Mathematics, vol. 182. Springer-Verlag, New York. Translated by Russell Thompson from the sixth edition of *Gewöhnliche Differentialgleichungen*, 1996.

Weihrauch, K. 2000. *Computable Analysis: An Introduction*. Texts in Theoretical Computer Science. Springer-Verlag.

Yao, A. C. 2003. Classical physics and the Church–Turing Thesis. *Journal of the Association for Computing Machinery 50,* 1, 100–105.

Received August 2007; accepted December 2007