CSC412 / CSC2506 Solutions to Sample Problems for Midterm

1. Let $p(k)$ be a one-dimensional discrete distribution that we wish to approximate, with support on non-negative integers. One way to fit an approximating distribution $q(k)$ is to minimize the Kullback-Leibler divergence:

$$KL(p||q) = \sum_{k=0}^{\infty} p(k) \log \frac{p(k)}{q(k)}$$

Show that when $q(k)$ is a Poisson distribution,

$$q(k) = \frac{\lambda^k \exp(-\lambda)}{k!}$$

this KL divergence is minimized by setting $\lambda$ to the mean of $p(k)$.

Solution:

$$\frac{\partial KL}{\partial \lambda} = 0 \implies \lambda = E[p(k)]$$

2. Recall that the definition of an exponential family model is:

$$f(x|\eta) = h(x)g(\eta)\exp(\eta^\top T(x))$$

where:

- $\eta$ are the parameters
- $T(x)$ are the sufficient statistics
- $h(x)$ is the base measure
- $g(\eta)$ is the normalizing constant

Consider the univariate Gaussian, with mean $\mu$ and precision $\lambda = \frac{1}{\sigma^2}$:

$$p(D|\mu, \lambda) = \prod_{i=1}^{N} \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left(-\frac{1}{2}(x_i - \mu)^2\right)$$

What are $\eta$ and $T(x)$ for this distribution when represented in exponential family form?

Solution:

$$p(D|\mu, \lambda) = (2\pi)^{-N/2}[\lambda^{1/2} \exp(-\lambda/2 \mu^2)]^N \exp[\mu \lambda \sum_i x_i - \lambda/2 \sum_i x_i^2]$$

$$\eta = \begin{bmatrix} \frac{\mu \lambda}{-\lambda/2} \end{bmatrix}$$

$$T(x) = \begin{bmatrix} \sum_i x_i \\ \sum_i x_i^2 \end{bmatrix}$$
3. Consider the following directed graphical model:

(a) List all variables that are independent of $A$ given evidence on $B$.
(b) Write down the factorized normalized joint distribution that this graphical model represents.

Solution: a) \{

4. Murphy 20.1
Correction! Older copies of the Murphy book have a typo pointing you to an incorrect figure.
Do this question but with this MRF:

Solution:

a) Largest intermediate term has size 3 (1,2,3) and (2,3,4)

b) Largest maximal clique has size 3.

c) The largest intermediate term has size 4 (2,3,4,5)

d) Largest maximal clique has size 4.
5. Consider the Factor Graph:

(a) Write down the normalized joint distribution \( P(X_1, X_2, X_3, X_4, X_5) \) in terms of the potentials.

(b) Write down any conditional independence relationships given by the graph.

Solution:

a) \( \frac{1}{Z} \phi_A(X_1) \phi_B(X_2) \phi_C(X_1, X_2, X_3) \phi_D(X_3, X_4) \phi_E(X_5) \)
where \( Z = \sum_X \phi_A(X_1) \phi_B(X_2) \phi_C(X_1, X_2, X_3) \phi_D(X_3, X_4) \phi_E(X_3, X_5) \)

b) \( X_1, X_2 \perp X_4, X_5 | X_3 \)
\( X_4 \perp X_5 | X_3 \)