CSC373— Algorithm Design, Analysis, and Complexity — Spring 2018

Tutorial Exercise 8: Self reduction of Vertex-Cover

A polynomial reduction of the VERTEX-COVER search problem to the decision problem is as follows.

```
def S = findMinVC(G)
    # Given undirected graph G = (V, E) find a minimum sized vertex cover S.
    # A polynomial reduction using VERTEX-COVER.
    # Input size: |s| = |V| + |E|.

    # Step 1: Use binary search to solve for minimum size k* of a vertex cover.
    k* = min{k ∈ [0, |V| ] | VERTEX-COVER(G, k) is true}
    # Requires O(log(|V|)) calls to VERTEX-COVER.

    S ← {}
    if k* == 0: # Input graph G must have no edges.
        return S

    V_r ← V; E_r ← E # "Remaining" graph to find a min cover for.
    for v in V:
        # Loop Invariant: All edges in E\E_r are covered by vertices in S, and
        #    G_r = (V_r, E_r) has a VERTEX-COVER of minimum size k* − |S|.

        k ← k* − |S|
        # We must cover all edges in the remaining graph G_r with only k vertices.

        # Build subgraph G' of G_r, with v and all edges terminating at v removed.
        V' ← V_r\{v}
        E_{r,v} ← {e ∈ E_r | e has endpoint v }
        E' ← E_r\E_{r,v}
        G' ← (V', E')

        if VERTEX-COVER(G', k − 1):
            # We can safely add v to S, since G' can be covered in k − 1 vertices
            S ← S ∪ {v}
            if |S| == k*
                break
            V_r ← V'
            E_r ← E'
    Assertion: |S| == k*
    return S
```

Proof of correctness. A simple inductive proof shows the loop invariant must be true. Details omitted. The remaining issue is that to show that when the loop terminates, the assertion at the end is true. That is, is it true that the loop can only exit with the break statement, rather than simply having the for loop exhausting the list of vertices? Note the loop cannot terminate with $|S| > k^*$ (since $|S|$ grows by only one each time the body of the if statement is executed).

1. Question: Complete the proof sketch. Either complete the sketch of this correctness proof or provide a counter-example that shows the above algorithm is incorrect. **Hint:** Consider enhancing the loop invariant.