CSC373— **Algorithm Design, Analysis, and Complexity — Spring 2018**

Tutorial Exercise 7: Circulations and Polynomial Reductions

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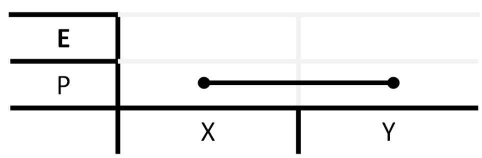
1. **Existence of a Circulation.** On slide 7 of the lecture notes on circulations and flows we stated:

   > **Characterization.** Given a circulation problem (V, E, c, d) there does **not** exist a circulation iff there exists a partition (A, B) of the vertices V (i.e., with $B = V \backslash A$) such that $\sum_{v \in B} d(v) > \text{cap}(A, B)$.

   The capacity of this partition (A, B) is defined to be $\text{cap}(A, B) \equiv \sum_{e \in A2B} c(e)$, where $A2B \equiv \{e \in E \mid e = (u, v), u \in A, v \in B\}$. Note this is similar to the capacity of s-t cuts except, for circulations, A and B are not restricted to contain s and t, respectively. Also, note that $\text{cap}(A, B)$ is not generally equal to $\text{cap}(B, A)$.)

2. **Picturing What Poly-Reductions Tell Us.** Suppose $X(s)$ and $Y(s)$ are two decision problems. We would like to know if these problems have a polynomial time solution or are they inherently exponential? That is, can $X(s)$ be solved with a deterministic algorithm that runs in $O(|s|^q)$ time, for some constant q? That is, is $X(s) \in P$? Or is it the case that, for every constant $q > 0$, the worst case runtime of any deterministic algorithm for solving $X(s)$ is necessarily $\Omega(|s|^q)$? We call this latter class E, for exponential.

   (a) What do we learn about the possible classifications of X and Y if we show $X \leq_p Y$? For example, it could be the case that both X and Y have polytime solutions. We can depict this pair of possibilities as the line segment in the figure below.

   Draw all other possible pairs in the figure above. Are any pairs ruled out? Explain.

   (b) In the lectures we showed

   $$\text{3-SAT} \leq_p \text{INDEPENDENT-SET} \equiv_p \text{VERTEX-COVER} \leq_p \text{SET-COVER}$$

   If any of these turn out to have a polytime solution, which others must also have polytime solutions? Similarly, if any of these are shown to be in E, as defined above, which others must be in E? Briefly explain.

3. **Set Packing.** The set packing decision problem is defined in a similar way to the definition provided below (we have only replaced the notion of a collection/family/list of subsets F with a simple set of subsets F):

   SetPack: Given a universe set U, a set of subsets $F = \{S_j \mid S_j \subseteq U, \ 1 \leq j \leq m\}$, and an integer k, does there exist $C \subseteq F$ with $|C| \geq k$ such that no two distinct elements $S_i, S_j \in C$ intersect (i.e., for all S_i, S_j in C with $S_i \neq S_j$ we have $S_i \cap S_j = \emptyset$)?

   (a) Denote the independent set decision problem by **IndepSet**. Show **IndepSet** $\leq_p$ **SetPack**.

   (b) Show **SetPack** $\leq_p$ **IndepSet**.

   (c) Define **searchSetPack** to be the search problem for set packing. That is, given U and F as in the Set-Packing decision problem, **find a subset** $C \subseteq F$ such that $|C|$ is the maximum possible and no two distinct elements in C intersect.

   Prove that **searchSetPack** $\leq_p$ **SetPack**.

   Hint 1: You need to first find k^*, the maximum possible size $|C|$. Then find the elements of C.

   Hint 2: For finding the elements of C it is useful to write a loop invariant stating that the current solution "is promising".