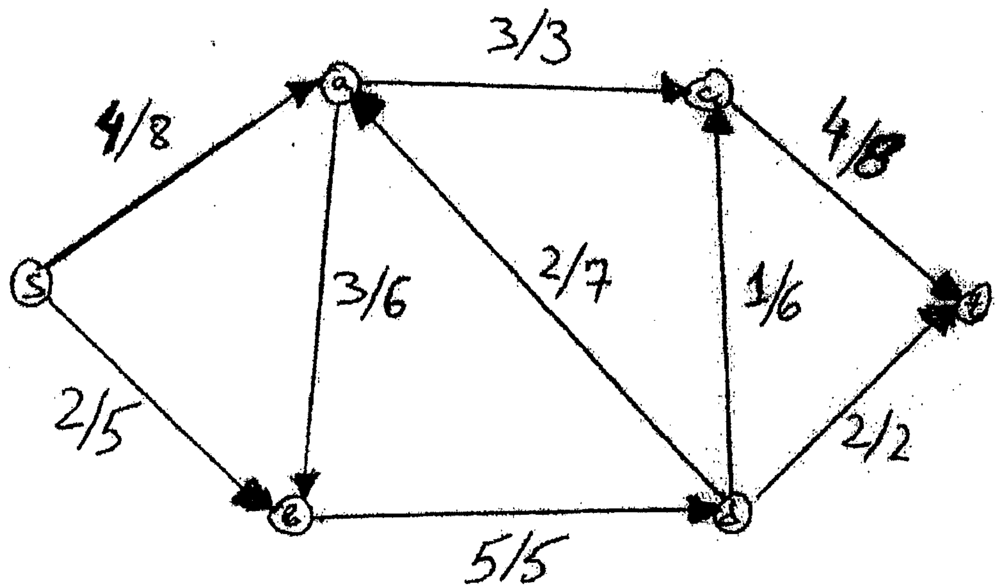
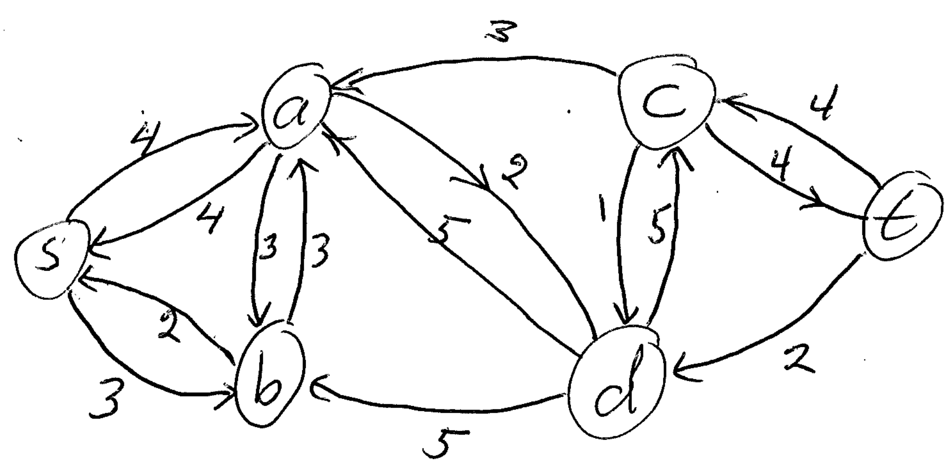
Question 1. (CONTINUED)

Part (c) [6 MARKS]

Consider the following flow network G with a flow f on it. On each edge e we write $f(e)/c(e)$ the flow through e and its capacity.



Compute the residual graph G_f and indicate an augmenting path on it.

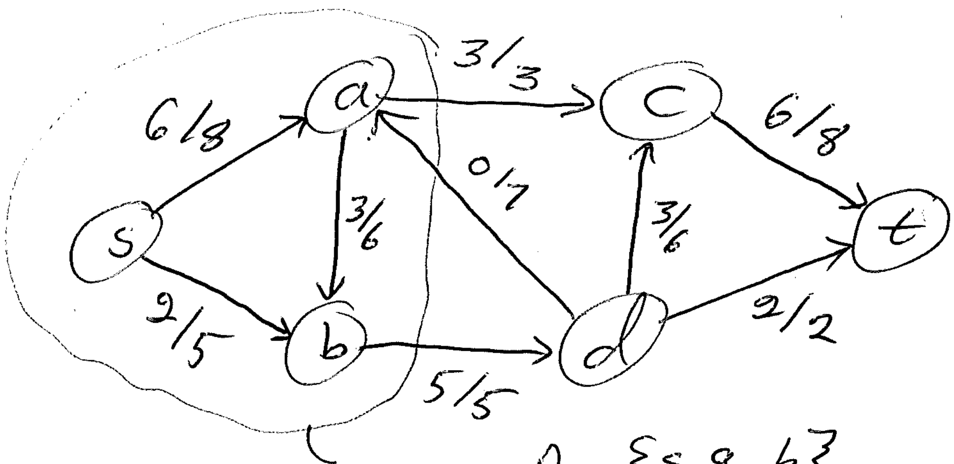


(s, a, d, c, t) is an augmenting path.

Part (d) [6 MARKS]

Do one step of Ford-Fulkerson's algorithm on G using the above augmenting path. Draw the resulting flow below.

If your solution is correct, you will obtain a max-flow in G. Show that this is indeed a max-flow, by exhibiting an appropriate cut in G.



$A = \{s, a, b\}$

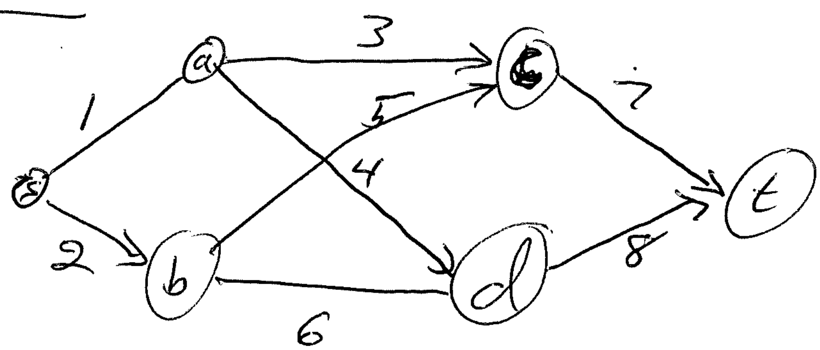
Flow 8, A set, $cap(A, V \setminus A) = 3 + 5 = 8$.

Part (e) [6 MARKS]

Decide whether the following statement is true or false. If it is true, give a brief explanation why. If it is false, give a counterexample.

Let G be a flow network such that all the capacities in G are distinct. Then there is a unique max-flow in G.

False



Max flow value 3. ~~Can alt~~

Can have $f((c,t)) = 3$ & $f((d,t)) = 0$

Or $f((c,t)) = 0$ & $f((d,t) = 3$, or ...

Question 5. (Applications of Network Flows) [20 MARKS]

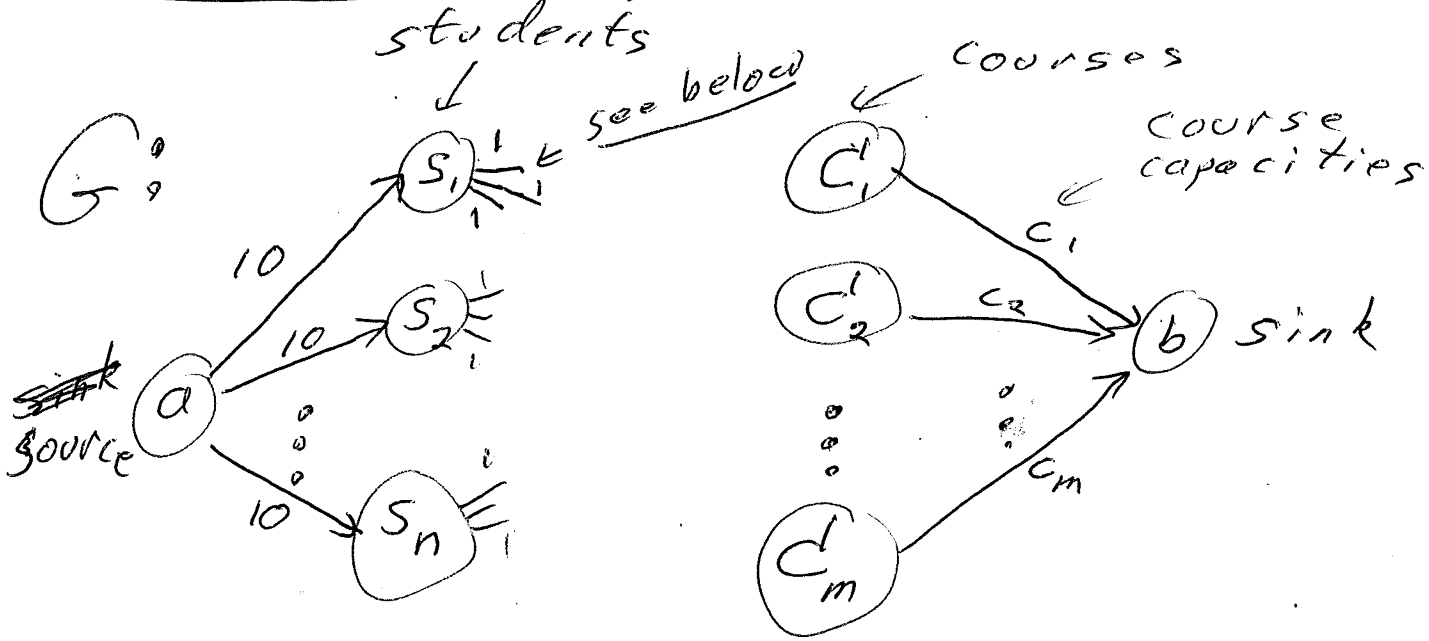
Consider the following problem of allocating students to classes. There are n students $s_1, s_2, \ldots, s_n$ and m classes $C_1, C_2, \ldots, C_m$. Each class C_j has its maximum enrollment capacity c_j. Each student s_i has a list L_i of 12 courses she would like to take during the next year. The university should enroll the student into 10 out of the 12 requested classes. The goal is to find an enrollment of the students to classes that satisfies the following conditions:

- the number of students in each class C_j does not exceed the capacity c_j;
- each student s_i is enrolled into exactly 10 classes from the list L_i.

Part (a) [10 MARKS]

Show how to convert this problem into a Max-Flow problem on a flow network.

Name source a, sink b. (too many s's)



For each student s_i there are 12 capacity one edges (s_i, C_k), where C_k is one of the 12 courses student s_i likes.

Find the max-flow of G.

Part (b) [4 MARKS]
Under what conditions on the max-flow on the network from part (a) is it possible to give an enrollment as required by the problem? Give a brief explanation.

Possible iff value of max flow equals ~~than~~ 10n. (Each (source, S_i) edge must be saturated.)
One unit of flow indicates that one student has been assigned to one course.
So a flow with value 10n is required for n students to be assigned 10 courses each.

Part (c) [6 MARKS]
Assume that the enrollment as required by the problem exists. Show how to obtain it from a max-flow on the network from part (a).
Explain why this enrollment satisfies the conditions. Let f be ~~the~~ a max-flow

For each S_i, assign S_i to courses
$$A_i \equiv \{ C_k \mid f((s_i, c_k)) = 1 \}.$$
As noted in (b), $f((\text{source}, S_i))$ must be 10.
By conservation at S_i, and the capacity limit of 1 on (S_i, C_k) edges, $|A_i| = 10$, as required.
students in course $C_k = \sum_{i=1}^{n} f((S_i, C_k))$
By conservation ~~this~~ at C_k, this equals
$f((C_k, \text{sink})) \underbrace{\leq c_k}$ $\equiv$ capacity course C_k.
The last inequality follows since f is a feasible flow.

Total Marks = 120