CSC373— Algorithm Design, Analysis, and Complexity — Spring 2018

Tutorial Exercise 9: NP-completeness of K-D Matching

---

1. **3D Matching.** We consider the following types of 3D matching problems.

   partial3DM: Given three distinct sets X, Y, and Z, with $|X| = |Y| = |Z| = n$, a set of triples $T \subseteq X \times Y \times Z$, and an integer k, does there exist a subset of triples $C \subseteq T$ of size $|C| \geq k$ such that no two distinct elements $C_i, C_j \in C$ have any element in common (i.e., if $C_i = (C_{i,1}, C_{i,2}, C_{i,3})$ and $C_j = (C_{j,1}, C_{j,2}, C_{j,3})$ are distinct triples in C then, for each $p = 1, 2, 3$, we have $C_{i,p} \neq C_{j,p}$)?

   perfect3DM: The input to this problem is similar to partial3DM except no integer k is provided. Instead, the question is whether there exists a set of triples $C \subseteq T$ such that $|C| = n$ and no two distinct elements $C_i, C_j \in C$ have any of their three elements in common? (That is, the matching is perfect in the sense that each element of X, Y, or Z is covered by exactly one triple in C.)

   Note Wikipedia defines the "3D matching problem" to be the problem partial3DM above, while the Kleinberg and Tardos text defines it to be perfect3DM. Moreover, in your answers below you can use the fact that 3-SAT $\leq_p$ perfect3DM, which is proved in the Kleinberg and Tardos text.

   (a) Show partial3DM is in NP.

   (b) Show perfect3DM $\leq_p$ partial3DM.

   (c) Given all the above results (including those quoted from Kleinberg and Tardos) can you conclude that partial3DM is NP-complete? Explain.

2. Consider partial2DM, which is the same as the 3D version except there are only two sets, X and Y, and T consists of a set of pairs $T \subseteq X \times Y$.

   (a) Is partial2DM in NP? Explain.

   (b) Show partial2DM $\leq_p$ SetPack, where SetPack is the set packing problem considered in the previous tutorial.

   (c) Is partial2DM NP-complete? Explain.

3. Consider partial4DM, which is the same as the 3D version except there are now four sets, W, X, Y and Z, while T consists of a set of 4-tuples $T \subseteq W \times X \times Y \times Z$.

   (a) Is partial4DM in NP? Explain.

   (b) Is partial4DM NP-complete? Explain.

4. **(Harder.)** Show partial3DM $\leq_p$ SAT by using an encoding of the constraints for 3D matching in terms of a CNF formula. Use the binary variables x_i, where x_i is true iff the i^{th} triple in T is to be included in the set C. (Note that we are asking simply for a reduction to SAT, not to 3-SAT.)