CSC373— Algorithm Design, Analysis, and Complexity — Spring 2018

Tutorial Exercise 11: Approximation Algorithms

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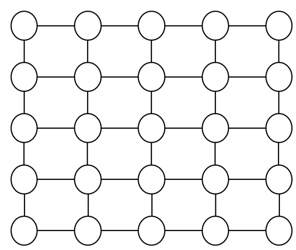
1. **Q5, Chp 11, Kleinberg and Tardos.** The first algorithm presented in the lecture notes for Load Balancing is an on-line algorithm, that is, the jobs can be processed as soon as they arrive. We refer to it as the **FirstArrival** algorithm. While the LPT algorithm obtains a better approximation ratio, it does not have this on-line property. Here we show that the **FirstArrival** algorithm can have an approximation ratio that is less than the worst case of 2 if the mix of jobs that it is asked to schedule is somehow restricted.

   For example, suppose that you have 10 machines, and need to schedule N jobs, where the n^{th} such job takes time t_n. In addition, you know the total time required for all the jobs is $T = \sum_{n=1}^{N} t_n = 3000$, and the time required for each individual job satisfies $1 \leq t_n \leq 50$. Show that the approximation ratio of FirstArrival for this mix of jobs is no worse than 7/6.

2. **Q10, Chp 11, Kleinberg and Tardos.** Suppose you are given a weighted graph $G = (V, E, w)$ where G has the form of an $n \times n$ grid graph (see figure below). Assume the weights $w(v)$ are non-negative integers.

   Prof. Jot proposes the following greedy algorithm for obtaining an approximate solution to the maximally weighted independent set problem for this type of graph:

   [S] = wIndSet(V, E, w)
   Initialize $F \leftarrow (V, E)$ and $S \leftarrow \{\ \}$.
   While the graph F is not empty:
       Find a vertex u in F with the largest weight $w(u)$.
       $S \leftarrow S \cup \{u\}$
       Update F by deleting the vertex u and all its neighbouring vertices v (i.e., all vertices v with an edge (u, v) still in F), and delete all the edges ending at any of these deleted vertices.
   End while
   return S

   (a) Write a loop invariant for the above code that is useful for proving that the set S returned by `wIndSet` is an independent set for the graph G.

   (b) Prove the loop invariant in part (a), and that the returned set S is an independent set for the graph G.

   (c) Show that $w(S) = \sum_{v \in S} w(v)$ is at least $(1/4)w(S^*)$, where S^* is an independent set of G with the maximum possible weight $w(S^*)$.

3. **Modified Q3 Chp 11 Kleiberg and Tardos.** Suppose you are given a list of N integers $L = [a_1, a_2, \ldots, a_N]$, and a positive integer C. The problem is to find a subset $S \subseteq \{1, 2, \ldots, N\}$ such that

   $$T(S) = \sum_{i \in S} a_i \leq C, \tag{1}$$

   and $T(S)$ is as large as possible.

(a) Prof. Jot proposes the following greedy algorithm for obtaining an approximate solution to this maximization problem:

> $[S] = \text{maxBoundedSetSum}([a_1, \ldots, a_N],\, C)$
> Initialize $S \leftarrow \{\ \}$, $T = 0$
> For $i = 1, 2, \ldots, N$:
>   If $T + a_i \leq C$:
> $S \leftarrow S \cup \{i\}$
> $T \leftarrow T + a_i$
> End for
> return S

Show that Prof. Jot's algorithm is not a ρ-appoximation algorithm for any fixed value ρ. (Use the convention that $\rho > 1$.)

(b) Describe a 2-approximation algorithm for this maximization problem that runs in $O(N \log(N))$ time.