Duality in Linear Programming

Learning Goals.

- Introduce the Dual Linear Program.
- Widget Example and Graphical Solution.
- Basic Theory:
  - Mutual Bound Theorem.
  - Duality Theorem.

Readings: Read text section 11.6, and sections 1 and 2 of Tom Ferguson's notes (see course homepage).

Standard Form for Linear Programs: Review

Consider a real-valued, unknown, n-vector $x = (x_1, x_2, \ldots, x_n)^T$.

A linear programming problem in standard form (A, b, c) has the three components:

Constants: A an m x n matrix, b an m x 1 vector, c an n x 1 vector.

Objective Function: We wish to choose x to maximize:

$c^T x = c_1x_1 + c_2x_2 + \ldots c_nx_n$ ← Linear function of x

with x subject to the following constraints:

Problem Constraints: For an m x n matrix A, and an m x 1 vector b:

$A x \le b$

Non-negativity Constraints:

$x \ge 0$

Linear inequality constraints on x

Notation: For two K-vectors x and y, $x \le y$ iff $x_k \le y_k$ for each k = 1, 2, …, K. Other inequalities ($\ge$, etc.) defined similarly.

Widget Factory Example: Revisited

Widget problem in Standard Form, constants (A, b, c).

Unknowns:
$x = (x_1, x_2)^T$ number (in thousands) of the two widget types.

Objective function (profit):
$c^T x = c_1x_1 + c_2x_2 = x_1 + 2x_2$, so $c^T = (c_1, c_2) = (1, 2)$.

Problem Constraints: $A x \le b$

$$\begin{aligned} x_1 + x_2 &\le 4, \\ -x_1 + x_2 &\le 1, \\ -3x_1 + 10x_2 &\le 15. \end{aligned} \quad \text{so} \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -3 & 10 \end{pmatrix}, \; b = \begin{pmatrix} 4 \\ 1 \\ 15 \end{pmatrix}$$

Non-negativity Constraints:

$x \ge 0$

Widget Factory Example: Upper Bounds

Maximize profit: $c^T x$, where $c^T = (c_1, c_2) = (1, 2)$.

Subject to: $Ax \le b$ and $x \ge 0$.

Notice, for any feasible x and any $y = (y_1, y_2, y_3)^T \ge 0$:

$$y^T A x = y^T \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -3 & 10 \end{pmatrix} x \le y^T b = y^T \begin{pmatrix} 4 \\ 1 \\ 15 \end{pmatrix}.$$

Could choose $y^T A \ge c$ and $y \ge 0$.

E.g., $y = (2, 0, 0)^T$ gives $y^T A = (2, 2) \ge c^T$. Therefore:

$c^Tx \le y^TAx \le y^Tb = 2b_1 = 8$, i.e. max profit $c^Tx \le 8$. ← Upper bound!

In general, for any feasible x and any y such that

$y \ge 0$ and $y^T A \ge c^T$, ← Feasible y

we have the inequality:

$c^Tx \le y^TAx \le y^Tb$. ← minimize y^Tb

Dual LP

Duality in Linear Programming

Defn. Consider the linear programming problem (in standard form):

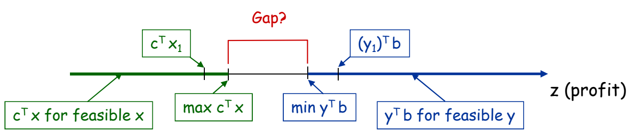
maximize $c^T x$
subject to $A x \le b$ and $x \ge 0$, (1)

The dual of this LP problem is the LP minimization problem:

minimize $y^T b$
subject to $y^T A \ge c^T$ and $y \ge 0$. (2)

These two LP problems are said to be duals of each other.

Mutual Bound Theorem: If x is a feasible solution of LP (1) and y is a feasible solution of LP (2), then $c^T x \le y^T A x \le y^T b$.
Pf: See previous slide.



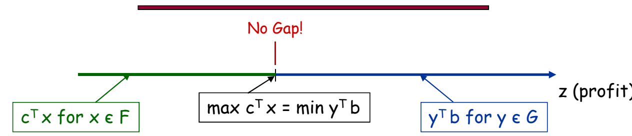
Duality Theorem of Linear Programming

LP Duality Theorem: Consider the linear programming problem:

maximize $c^T x$
subject to $A x \le b$ and $x \ge 0$. (1)

The feasible set F for (1) is not empty and $c^T x$ is bounded above for $x \in F$ iff the corresponding dual LP (2) (above) has a non-empty feasible set $G = \{y \mid y^T A \ge c^T \text{ and } y \ge 0\}$ and $y^T b$ is bounded below for $y \in G$. Moreover, in this case, $\max \{ c^T x \mid x \in F \} = \min \{ y^T b \mid y \in G \}$.

Note: For integer linear programming (i.e., $x_i, y_j \in \mathbb{Z}$) there can be a gap.



Vertices of Dual Linear Program

Consider the Dual LP problem:

minimize $y^T b$
subject to $y^T A \ge c^T$ and $y \ge 0$. (2)

We can rewrite the feasibility conditions (2) of the dual as

m x (n + m) matrix

$$y^T D \equiv y^T \begin{pmatrix} A & I \end{pmatrix} \ge d^T \equiv \begin{pmatrix} c^T & 0^T \end{pmatrix}. \qquad (3)$$

The dual LP is an LP, and vertices can be defined the same way as we did before.

Let $t = \{t_1, t_2, \ldots, t_m\}$ be a selection of m columns of (3), $1 \le t_i \le m+n$. Define E(t) to be the m x m matrix formed from the t-columns of D, and $e^T(t)$ the (1 x m)-vector formed from the same columns of d^T.

A point $v \in \mathbb{R}^m$ is a vertex of the feasible set (3) iff there exists an t such that E(t) is nonsingular, $v^T = e^T(t)[E(t)]^{-1}$, and v satisfies (3).

Vertices of LP and Dual LP

Define the m+n dimensional binary valued indicator vector δ(s) where $\delta_j = 1$ if $j \in s$, and $\delta_j = 0$ otherwise. Define δ(t) similarly.

$$\delta(s) = (\alpha_1, \alpha_2, \ldots, \alpha_m, \quad \alpha_{m+1}, \ldots, \alpha_{m+n}),$$
$$\delta(t) = (\beta_1, \ldots, \beta_n, \quad \beta_{n+1}, \beta_{n+2}, \ldots, \beta_{n+m}).$$

Vertex of LP: If the j^{th} coefficient of δ(s) is one (i.e., $[\delta(s)]_j = 1$) then the j^{th} row below is an equality for vertex x:

$$Px \equiv \begin{pmatrix} A \\ -I \end{pmatrix} x \le p \equiv \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Vertex of Dual LP: If the i^{th} coefficient of δ(t) is one (i.e., $[\delta(t)]_i = 1$) then the i^{th} column below is an equality for vertex y:

$$y^T D \equiv y^T \begin{pmatrix} A & I \end{pmatrix} \ge d^T \equiv \begin{pmatrix} c^T & 0^T \end{pmatrix}.$$

Complementary Slackness

Complementary Slackness: Given feasible solutions x and y of the LP and the dual LP, respectively. Then x and y are optimal iff

$$\sum_{j=1}^{n} A_{i,j} x_j \; < \; b_i \;\; \text{implies} \;\; y_i = 0,$$

and

$$\sum_{i=1}^{m} y_i A_{i,j} \; > \; c_j \;\; \text{implies} \;\; x_j = 0.$$

Pf: Follows from $c^T x = y^T A x = y^T b$ as a necessary and sufficient condition for the optimality of the feasible solutions x and y.

Suggests choosing of the sets s and t (defining the vertex x of the LP and the vertex y of the dual LP) such that the bit vectors satisfy:

$$[\delta(s)]_i \;=\; \text{not } [\delta(t)]_{i+n},$$
$$[\delta(t)]_j \;=\; \text{not } [\delta(s)]_{j+m}.$$

Proposing Vertices for the Dual LP

Spatially, complementary slackness suggests:

$$\delta(s) \;=\; (\alpha_1, \alpha_2, \ldots, \alpha_m, \quad \alpha_{m+1}, \ldots, \alpha_{m+n}),$$

$$\delta(t) \;=\; (\beta_1, \ldots, \beta_n, \quad \beta_{n+1}, \beta_{n+2}, \ldots, \beta_{n+m}).$$

Where β_{i+n} = bitFlip(α_i) for i = 1, 2, …, m.
And β_j = bitFlip(α_{j+m}) for j = 1, 2, …, n.

Since sum(δ(s)) = n, length(δ(·)) = n+m, it follows sum(δ(t)) = m.

Given a vertex x of the LP, defined by s, we can use the rule above to try to construct t and the corresponding vertex of the dual LP. We can use the pair to check for optimality. See the following example.

Graphing the Widget Factory Example: Cont.

Example: $x = (x_1, x_2)^T$. Linear Program specified by (A, b, c).

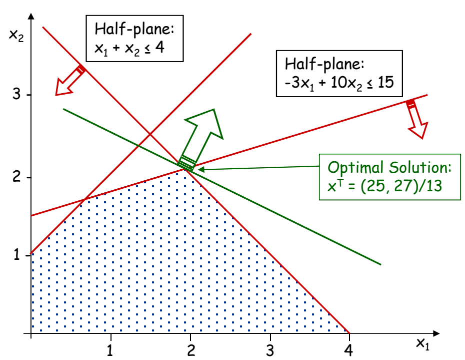
Objective Function: $c^T x$,
$c = (1, 2)^T$

Problem Constraints:
$Ax \le b$,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -3 & 10 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 1 \\ 15 \end{pmatrix}.$$

Non-negativity:
$x \ge 0$.

Optimal Vertex:
s = { 1, 3 }



Widget Factory Example: Optimal Dual Solution

E.G. (Cont.): This vertex of the LP was obtained using s = {1, 3}. Generate corresponding column selection t (possibly a feasible vertex for dual LP):

$$\delta(s) \;=\; (1, 0, 1, \; 0, 0)$$

$$\delta(t) \;=\; (1, 1, \; 0, 1, 0)$$

So t = { 1, 2, 4} and we select columns 1, 2, and 4 from (3) below.

$$y^T (A \;\; I) \;\ge\; (c^T \; 0^T). \qquad (3)$$

$$y^T \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ -3 & 10 & 0 \end{pmatrix} \;=\; (1 \; 2 \; 0).$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -3 & 10 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 1 \\ 15 \end{pmatrix}, \quad c^T = (1 \; 2).$$

Soln: $y^T = (16, 0, 1)/13$.

Check: $c^T x = (1, 2)\,(25, 27)^T/13 = 79/13 = y^T b = (16, 0, 1)/13\,(4, 1, 15)^T$

Conclude: y is a feasible vertex of the dual LP, and x, y are optimal.