# Linear Subspace Models

**Goal:** Explore linear models of a data set.

**Motivation:**

A central question in vision concerns how we represent a collection of data vectors. The data vectors may be rasterized images, for example.

- We consider the construction of low-dimensional bases for an ensemble of training data using principal components analysis (PCA).
- We introduce PCA, its derivation, its properties, and some of its uses.
- We very briefly critique its suitability for object detection.

**Readings:** Sections 22.1–22.3 of the Forsyth and Ponce.

**Matlab Tutorials:** colourTutorial.m, trainEigenEyes.m and detectEigenEyes.m

# Representing Images of Human Eyes

**Question:** Suppose we have a dataset of scaled, centered images of human eyes. How can we find an efficient representation of such a data set?

Left Eyes Right Eyes

**Generative Model.** Suppose we can approximate each image in the data set with a parameterized model of the form,

$$I(\vec{x}) \approx g(\vec{x}; \vec{a}).$$

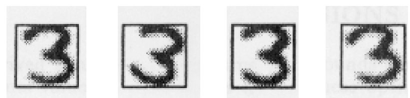
Here $\vec{a}$ is a vector of coefficients.

**Possible uses:**

- compress or reduce the dimension of the data set,
- generate novel instances,
- (possibly) recognition.

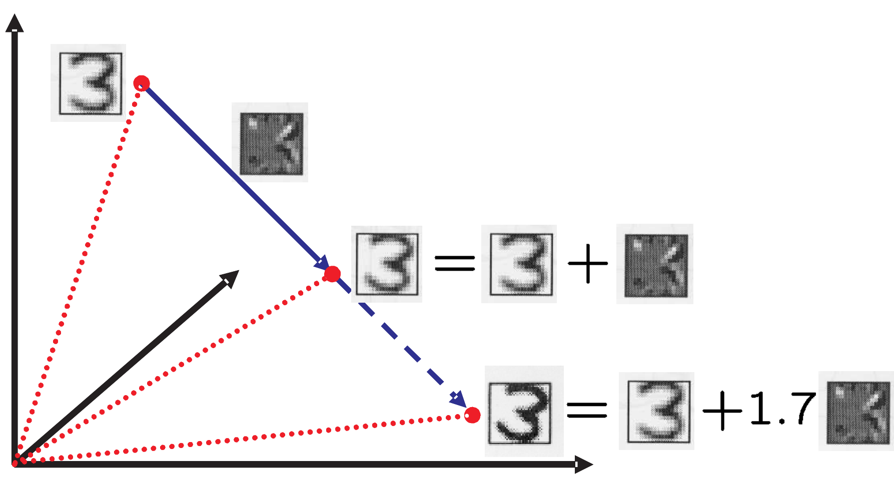
# Subspace Appearance Models

**Idea:** Images are not random, especially those of an object, or similar objects, under different viewing conditions.



Rather, than storing every image, we might try to represent the images more effectively, e.g., in a lower dimensional *subspace*.

For example, let's represent each $N \times N$ image as a point in an $N^2$-dim vector space (e.g., ordering the pixels lexicographically to form the vectors).




(red points denote images, blue vectors denote image differences)

How do we find a low-dimensional basis to accurately model (approximate) each image of the training ensemble (as a linear combination of basis images)?

# Linear Subspace Models

We seek a linear basis with which each image in the ensemble is approximated as a linear combination of basis images $b_k(\vec{\mathbf{x}})$

$$I(\vec{\mathbf{x}}) \;\approx\; m(\vec{x}) + \sum_{k=1}^{K} a_k\, b_k(\vec{\mathbf{x}}), \tag{1}$$

here $m(\vec{x}\,)$ is the mean of the image ensemble. The *subspace coefficients* $\vec{\mathbf{a}} = (a_1, ..., a_K)$ comprise the representaion.

With some abuse of notation, assuming basis images $b_k(\vec{\mathbf{x}})$ with $N^2$ pixels, let's define

- $\vec{\mathbf{b}}_k$ – an $N^2 \times 1$ vector with pixels arranged in lexicographic order
- $\mathbf{B}$ – a matrix with columns $\vec{\mathbf{b}}_k$, *i.e.*, $\mathbf{B} = [\vec{\mathbf{b}}_1, ..., \vec{\mathbf{b}}_K] \in \mathcal{R}^{N^2 \times K}$

With this notation we can rewrite Eq. (1) in matrix algebra as

$$\vec{\mathbf{I}} \;\approx\; \vec{m} + \mathbf{B}\,\vec{\mathbf{a}} \tag{2}$$

In what follows, we assume that the mean of the ensemble is $\vec{0}$. (Otherwise, if the ensemble we have is not mean zero, we can estimate the mean and subtract it from each image.)

## Choosing The Basis

**Orthogonality:** Let's assume orthogonal basis functions,

$$\| \vec{\mathbf{b}}_k \|_2 \; = \; 1 \quad , \qquad \vec{\mathbf{b}}_j^{\,T} \vec{\mathbf{b}}_k \; = \; \delta_{jk} \; .$$

**Subspace Coefficients:** It follows from the linear model in Eq. (2) and the orthogonality of the basis functions that

$$\vec{\mathbf{b}}_k^{\,T} \vec{\mathbf{I}} \; \approx \; \vec{\mathbf{b}}_k^{\,T} \mathbf{B}\, \vec{\mathbf{a}} \; = \; \vec{\mathbf{b}}_k^{\,T} \, [\vec{\mathbf{b}}_1, ..., \vec{\mathbf{b}}_K]\, \vec{\mathbf{a}} \; = \; a_k$$

This selection of coefficients, $\vec{\mathbf{a}} = \mathbf{B}^T \vec{\mathbf{I}}$, minimizes the sum of squared errors (or sum of squared pixel differences, SSD):

$$\min_{\vec{\mathbf{a}} \in \mathcal{R}^K} \| \, \vec{\mathbf{I}} - \mathbf{B}\, \vec{\mathbf{a}} \, \|_2^2$$

**Basis Images:** In order to select the basis functions $\{\vec{\mathbf{b}}_k\}_{k=1}^K$, suppose we have a training set of images

$$\{ \, \vec{\mathbf{I}}_l \, \}_{l=1}^L \quad , \quad \text{with} \;\; L \gg K$$

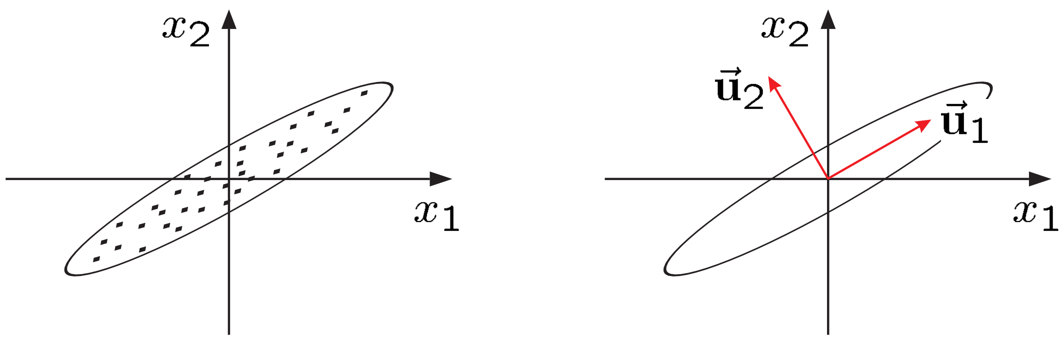
Recall we are assuming the images are mean zero.

Finally, let's select the basis, $\{\vec{\mathbf{b}}_k\}_{k=1}^K$, to minimize squared reconstruction error:

$$\sum_{l=1}^{L} \min_{\vec{\mathbf{a}}_l} \| \, \vec{\mathbf{I}}_l - \mathbf{B}\, \vec{\mathbf{a}}_l \, \|_2^2$$

# Intuitions

**Example:** Let's consider a set of images $\{\, \vec{\mathbf{I}}_l \,\}_{l=1}^{L}$, each with only two pixels. So, each image can be viewed as a 2D point, $\vec{\mathbf{I}}_l \in \mathcal{R}^2$.




For a model with only one basis image, what should $\vec{\mathbf{b}}_1$ be?

**Approach:** Fit an ellipse to the distribution of the image data, and choose $\vec{\mathbf{b}}_1$ to be a unit vector in the direction of the major axis.

Define the ellipse as $\vec{\mathbf{x}}^T C^{-1} \vec{\mathbf{x}} = 1$, where $C$ is the sample covariance matrix of the image data,

$$\mathbf{C} = \frac{1}{L} \sum_{l=1}^{L} \vec{\mathbf{I}}_l \, \vec{\mathbf{I}}_l^{\,T}$$

The eigenvectors of $\mathbf{C}$ provide the major axis, i.e.,

$$\mathbf{C}\,\mathbf{U} \;=\; \mathbf{U}\,\mathbf{D}$$

for orthogonal matrix $\mathbf{U} = [\vec{\mathbf{u}}_1, \, \vec{\mathbf{u}}_2]$, and diagonal matrix $\mathbf{D}$ with elements $d_1 \geq d_2 \geq 0$. The direction $\vec{\mathbf{u}}_1$ associated with the largest eigenvalue is the direction of the major axis, so let $\vec{\mathbf{b}}_1 = \vec{\mathbf{u}}_1$.

# Principal Components Analysis

**Theorem:** *(Minimum reconstruction error)* The orthogonal basis $\mathbf{B}$, of rank $K < N^2$, that minimizes the squared reconstruction error over training data, $\{\vec{\mathbf{I}}_l\}_{l=1}^{L}$, i.e.,

$$\sum_{l=1}^{L} \min_{\vec{\mathbf{a}}_l} \| \vec{\mathbf{I}}_l - \mathbf{B}\,\vec{\mathbf{a}}_l \|_2^2$$

is given by the first $K$ eigenvectors of the data covariance matrix

$$\mathbf{C} = \frac{1}{L} \sum_{l=1}^{L} \vec{\mathbf{I}}_l \, \vec{\mathbf{I}}_l^{\,T} \;\in\; \mathcal{R}^{N^2 \times N^2} \;, \quad \text{for which} \quad \mathbf{C}\,\mathbf{U} \;=\; \mathbf{U}\,\mathbf{D}$$

where $\mathbf{U} = [\vec{\mathbf{u}}_1, ..., \vec{\mathbf{u}}_{N^2}]$ is orthogonal, and $\mathbf{D} = \text{diag}(d_1, ..., d_{N^2})$ with $d_1 \geq d_2 \geq \, ... \, \geq d_{N^2}$.

That is, the optimal basis vectors are $\vec{\mathbf{b}}_k = \vec{\mathbf{u}}_k$, for $k = 1...K$. The corresponding basis images $\{b_k(\vec{\mathbf{x}})\}_{k=1}^{K}$ are often called eigen-images.

**Proof:** see the derivation below.

# Derivation of PCA

To begin, we want to find $\mathbf{B}$ in order to minimize squared error in subspace approximations to the images of the training ensemble.

$$E \;=\; \sum_{l=1}^{L} \min_{\vec{\mathbf{a}}_l} \parallel \vec{\mathbf{I}}_l - \mathbf{B}\,\vec{\mathbf{a}}_l \parallel_2^2$$

Given the assumption that the columns of $\mathbf{B}$ are orthonormal, the optimal coefficients are $\vec{\mathbf{a}}_l = \mathbf{B}^T\vec{\mathbf{I}}_l$, so

$$E \;=\; \sum_{l=1}^{L} \min_{\vec{\mathbf{a}}_l} \parallel \vec{\mathbf{I}}_l - \mathbf{B}\,\vec{\mathbf{a}}_l \parallel_2^2 \;\;=\;\; \parallel \vec{\mathbf{I}}_l - \mathbf{B}\,\mathbf{B}^T\vec{\mathbf{I}}_l \parallel_2^2 \tag{3}$$

Furthermore, writing the each training image as a column in a matrix $\mathbf{A} = \left[\vec{\mathbf{I}}_1, ..., \vec{\mathbf{I}}_L\right]$, we have

$$E \;=\; \sum_{l=1}^{L} \parallel \vec{\mathbf{I}}_l - \mathbf{B}\,\mathbf{B}^T\vec{\mathbf{I}}_l \parallel_2^2 \;\;=\;\; \parallel \mathbf{A} - \mathbf{B}\,\mathbf{B}^T\mathbf{A} \parallel_F^2 \;\;=\; trace\left[\,\mathbf{A}\,\mathbf{A}^T\,\right] - trace\left[\,\mathbf{B}^T\mathbf{A}\,\mathbf{A}^T\mathbf{B}\,\right]$$

You get this last step by expanding the square and noting $\mathbf{B}^T\mathbf{B} = \mathbf{I}_K$, and using the properties of $trace$, e.g., $trace[\mathbf{A}] = trace[\mathbf{A}^T]$, and also $trace[\mathbf{B}^T\mathbf{A}\mathbf{A}^T\mathbf{B}] = trace[\mathbf{A}^T\mathbf{B}\mathbf{B}^T\mathbf{A}]$ .
So to minmize the average squared error in the approximation we want to find $\mathbf{B}$ to maximize

$$E' \;=\; trace\left[\,\mathbf{B}^T\mathbf{A}\,\mathbf{A}^T\mathbf{B}\,\right] \tag{4}$$

Now, let's use the fact that for the data covariance, $\mathbf{C}$ we have $\mathbf{C} = \frac{1}{L}\mathbf{A}\,\mathbf{A}^T$. Moreover, as defined above the SVD of $\mathbf{C}$ can be written as $\mathbf{C} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^T$. So, let's substitute the SVD into $E'$:

$$E' \;=\; trace\left[\,\mathbf{B}^T\mathbf{U}\,\mathbf{D}\,\mathbf{U}^T\mathbf{B}\,\right] \tag{5}$$

where of course $\mathbf{U}$ is orthogonal, and $\mathbf{D}$ is diagonal.

Now we just have to show that we want to choose $\mathbf{B}$ such that the trace strips off the first $K$ elements of $\mathbf{D}$ to maximize $E'$. Intuitively, note that $\mathbf{B}^T\mathbf{U}$ must be rank $K$ since $\mathbf{B}$ is rank $K$. And note that the diagonal elements of $\mathbf{D}$ are ordered. Also the trace is invariant under matrix rotation. So, the highest rank $K$ trace we can hope to get is by choosing $\mathbf{B}$ so that, when combined with $\mathbf{U}$ we keep the first $K$ columns of $\mathbf{D}$. That is, the columns of $\mathbf{B}$ should be the first $K$ orthonormal rows of $\mathbf{U}$. We need to make this a little more rigorous, but that's it for now...

# Other Properties of PCA

**Maximum Variance:** The $K$-D subspace approximation captures the greatest possible variance in the training data.

- For $a_1 = \vec{\mathbf{b}}_1^T \vec{\mathbf{I}}$, the direction $\vec{\mathbf{b}}_1$ that maximizes the variance $\mathrm{E}[a_1^2] = \vec{\mathbf{b}}_1^{\,T} \mathbf{C} \vec{\mathbf{b}}_1$, subject to $\vec{\mathbf{b}}_1^T \vec{\mathbf{b}}_1 = 1$, is the first eigenvector of $\mathbf{C}$.
- The second maximizes $\vec{\mathbf{b}}_2^T \mathbf{C} \vec{\mathbf{b}}_2$ subject to $\vec{\mathbf{b}}_2^T \vec{\mathbf{b}}_2 = 1$ and $\vec{\mathbf{b}}_1^T \vec{\mathbf{b}}_2 = 0$.
- For $a_k = \vec{\mathbf{b}}_k^{\,T} \vec{\mathbf{I}}$, and $\vec{\mathbf{a}} = (a_1, ..., a_K)$, the subspace coefficient covariance is $\mathrm{E}[\vec{\mathbf{a}}\,\vec{\mathbf{a}}^T] = \mathrm{diag}(d_1, ..., d_K)$. That is, the diagonal entries of $\mathbf{D}$ are marginal variances of the subspace coefficients:

$$\sigma_k^2 \;\equiv\; \mathrm{E}[a_k^2] \;=\; d_k \; .$$

  So the total variance *captured* in the subspace is sum of first $K$ eigenvalues of $\mathbf{C}$.

- Total variance *lost* owing to the subspace projection (i.e., the out-of-subspace variance) is the sum of the last $N^2 - K$ eigenvalues:

$$\frac{1}{L} \sum_{l=1}^{L} \left[ \min_{\vec{\mathbf{a}}_l} \| \, \vec{\mathbf{I}}_l - \mathbf{B} \, \vec{\mathbf{a}}_l \, \|_2^2 \right] \;=\; \sum_{k=K+1}^{N^2} \sigma_k^2$$

**Decorrelated Coefficients:** $\mathbf{C}$ is diagonalized by its eigenvectors, so $\mathbf{D}$ is diagonal, and the subspace coefficients are uncorrelated.

- Under a Gaussian model of the images (where the images are drawn from an $N^2$-dimensional Gaussian pdf), this means that the coefficients are also statistically independent.

# PCA and Singular Value Decomposition

The singular value decomposition of the data matrix $\mathbf{A}$,

$$\mathbf{A} = \left[\vec{\mathbf{I}}_1, ..., \vec{\mathbf{I}}_L\right] \ , \quad \mathbf{A} \in \mathcal{R}^{N^2 \times L} \ , \quad \text{where usually } \ L \ll N^2 \ .$$

is given by

$$\mathbf{A} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T$$

where $\mathbf{U} \in \mathcal{R}^{N^2 \times L}$ , $\mathbf{S} \in \mathcal{R}^{L \times L}$ , $\mathbf{V} \in \mathcal{R}^{L \times L}$ . The columns of $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, i.e., $\mathbf{U}^T\mathbf{U} = \mathbf{I}_{L\times L}$ and $\mathbf{V}^T\mathbf{V} = \mathbf{I}_{L\times L}$, and matrix $\mathbf{S}$ is diagonal, $\mathbf{S} = \text{diag}(s_1, ..., s_L)$ where $s_1 \geq s_2 \geq ... \geq s_L \geq 0$.

**Theorem:** The best rank-$K$ approximation to $\mathbf{A}$ under the Frobenius norm, $\tilde{\mathbf{A}}$, is given by

$$\tilde{\mathbf{A}} = \sum_{k=1}^{K} s_k \vec{\mathbf{u}}_k \vec{\mathbf{v}}_k{}^T = \mathbf{B}\,\mathbf{B}^T\mathbf{A} \ , \ \text{ where } \min_{rank(\tilde{\mathbf{A}})=K} \| \mathbf{A} - \tilde{\mathbf{A}} \|_F^2 = \sum_{k=K+1}^{N^2} s_k^2 \ ,$$

and $\mathbf{B} = [\vec{\mathbf{u}}_1, ..., \vec{\mathbf{u}}_K]$. $\tilde{\mathbf{A}}$ is also the best rank-$K$ approximation under the $L_2$ matrix norm.

What's the relation to PCA and the covariance of the training images?

$$\mathbf{C} = \frac{1}{L} \sum_{l=1}^{L} \vec{\mathbf{I}}_l \, \vec{\mathbf{I}}_l{}^T = \frac{1}{L} \mathbf{A}\,\mathbf{A}^T = \frac{1}{L} \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T\mathbf{V}\,\mathbf{S}^T\,\mathbf{U}^T = \frac{1}{L} \mathbf{U}\,\mathbf{S}^2\,\mathbf{U}^T$$

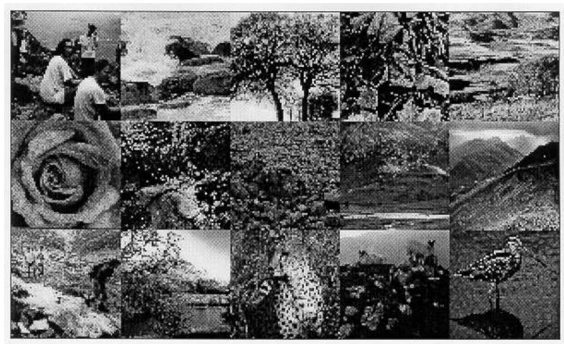
So the squared singular values of $\mathbf{A}$ are proportional to the first $L$ eigenvalues of $\mathbf{C}$:

$$d_k = \begin{cases} \frac{1}{L} s_k^2 & \text{for } k = 1, ..., L \\ 0 & \text{for } k > L \end{cases}$$

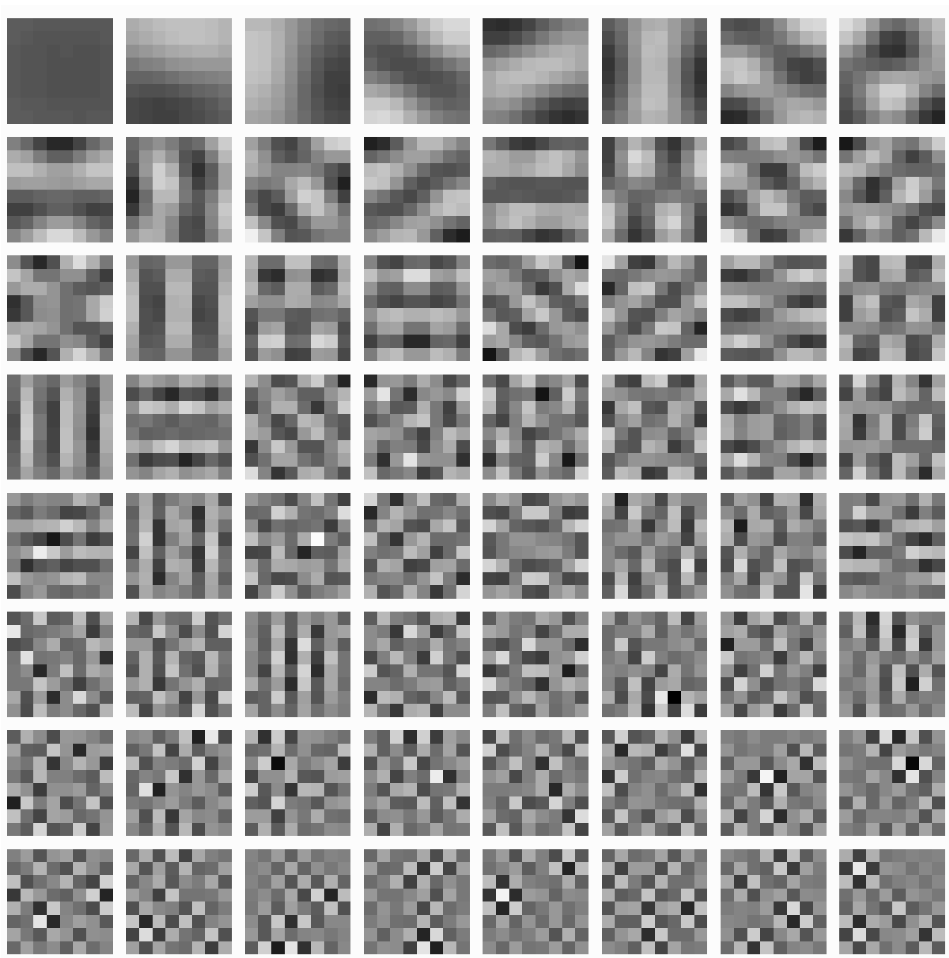
And the singular vectors of $\mathbf{A}$ are just the first $L$ eigenvectors of $\mathbf{C}$.

# Eigen-Images for Generic Images?

Fourier components are eigenfunctions of generic image ensembles.



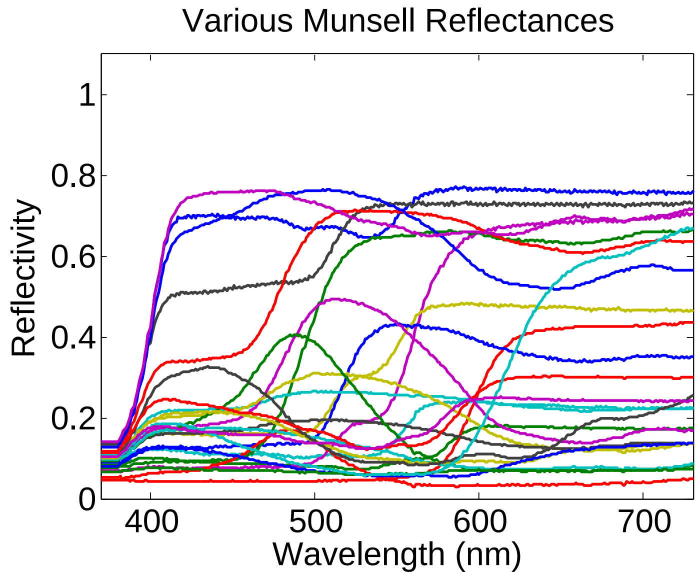
Why? Covariance matrices for stationary processes are Toeplitz.



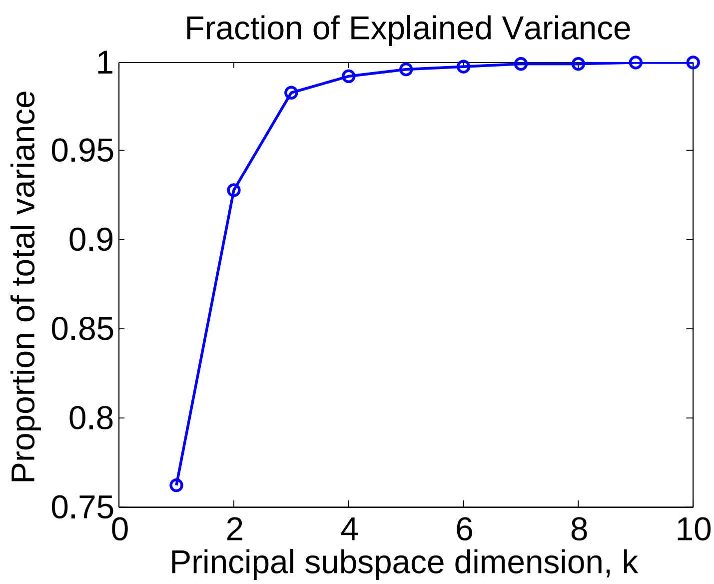
PCA yields unique eigen-images up to rotations of invariant subspaces (e.g., Fourier components with the same marginal variance).

# Eigen-Reflectances

Consider an ensemble of surface reflectances $r(\lambda)$.




What is the effective dimension of these reflectances? Define $V_k \equiv \sum_{j=1}^{k} \sigma_j^2$. Then the fraction of total variance explained by the first $k$ PCA components is $Q_k \equiv V_k/V_L$.



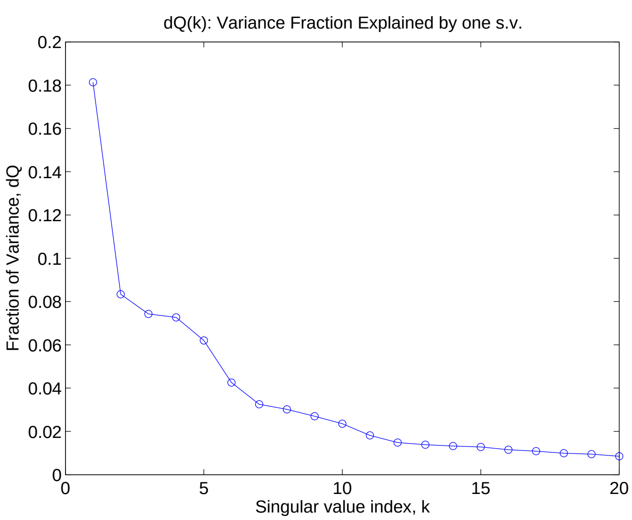

Reflectances $r(\lambda)$, for wavelengths $\lambda$ within the visible spectrum, are effectively 3 dimensional (see `colourTutorial.m`).

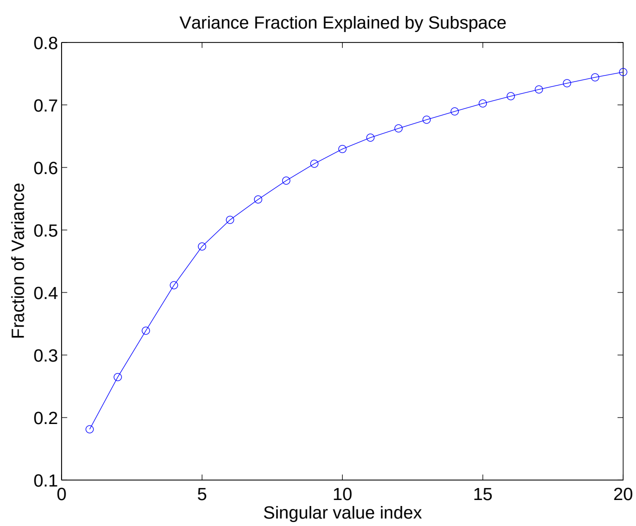
# Eye Subspace Model

**Subset of 1196 eye images ($25 \times 20$):**

Left Eyes Right Eyes

**Defn:** Let $V_k \equiv \sum_{j=1}^{k} s_j^2$, $dQ_k \equiv s_k^2 / V_L$, and $Q_k \equiv V_k / V_L$:



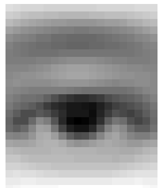




**Left** plot shows $dQ_k$, the fraction of the total variance contributed by the $k^{th}$ principal component.

**Right** plot shows $Q_k$ the fraction of the total variance captured by the subspace formed from the first $k$ principal components.

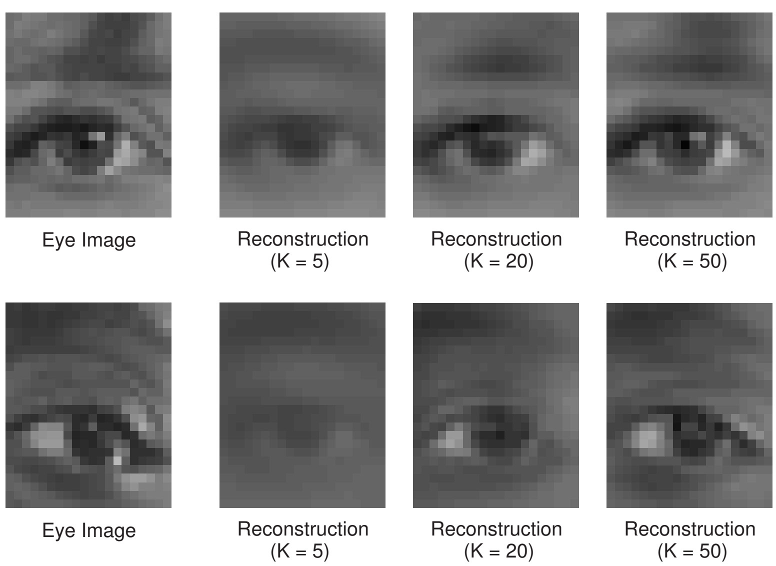
# Eye Subspace Model

**Mean Eye:**



**Basis Images** $(1-6, \text{ and } 10:5:35)$**:**

**Reconstructions** (for $K = 5, 20, 50$)**:**



# Generative Eye Model

Generative model, $\mathcal{M}$, for random eye images:

$$\vec{\mathbf{I}} = \vec{\mathbf{m}} + \left( \sum_{k=1}^{K} a_k \vec{\mathbf{b}}_k \right) + \vec{\mathbf{e}}$$

where $\vec{\mathbf{m}}$ is the mean eye image, $a_k \sim \mathcal{N}(0, \sigma_k^2)$, $\sigma_k^2$ is the sample variance associated with the $k^{th}$ principal direction in the training data, and $\vec{\mathbf{e}} \sim \mathcal{N}(0, \sigma_e^2 \, \mathbf{I}_{N^2})$ where $\sigma_e^2 = \frac{1}{N^2} \sum_{k=K+1}^{N^2} \sigma_k^2$ is the per pixel out-of-subspace variance.

**Random Eye Images:**

Random draws from generative model (with K = 5, 10, 20, 50, 100, 200)

So the probability of an image given this model $\mathcal{M}$ is

$$p(\vec{\mathbf{I}} \,|\, \mathcal{M}) = \left( \prod_{k=1}^{K} p(a_k | \mathcal{M}) \right) p(\vec{\mathbf{e}} \,|\, \mathcal{M})$$

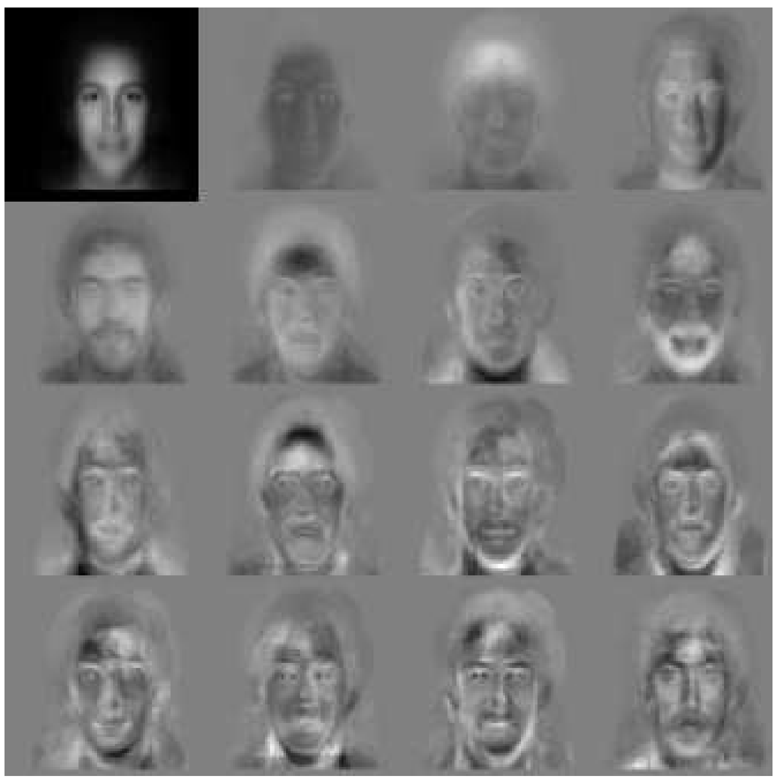
where

$$p(a_k|\mathcal{M}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{a_k^2}{2\sigma_k^2}} \quad , \qquad p(\vec{\mathbf{e}} \,|\, \mathcal{M}) = \prod_{j=1}^{N^2} \frac{1}{\sqrt{2\pi}\sigma_e} e^{-\frac{e_j^2}{2\sigma_e^2}} \quad .$$

# Face Detection

The wide-spread use of PCA for object recognition began with the work Turk and Pentland (1991) for face detection and recognition.

Shown below is the model learned from a collection of frontal faces, normalized for contrast, scale, and orientation, with the backgrounds removed prior to PCA.

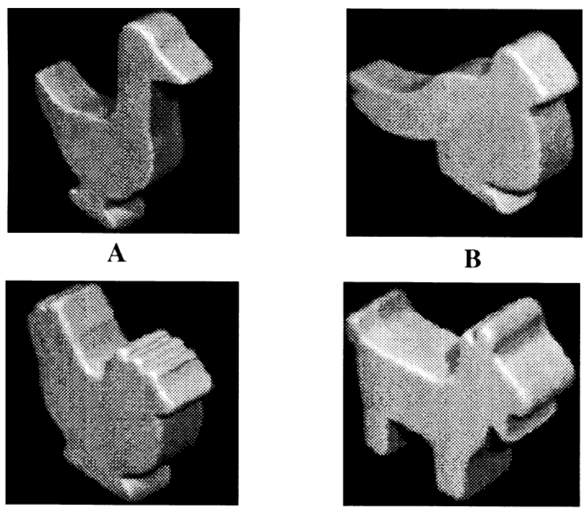


Here are the mean image (upper-left) and the first 15 eigen-images. The first three show strong variations caused by illumination. The next few appear to correspond to the occurrence of certain features (hair, hairline, beard, clothing, etc).

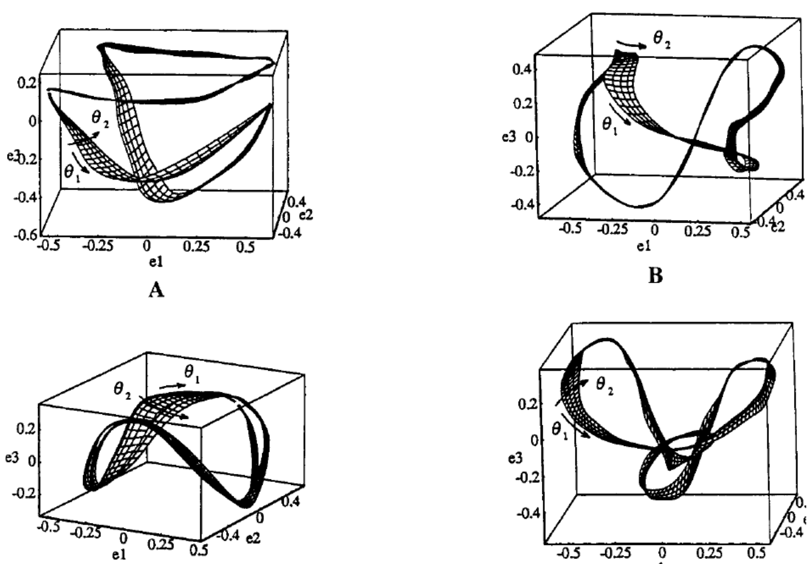
# Object Recognition

Murase and Nayar (1995)

- images of multiple objects, taken from different positions on the viewsphere
- each object occupies a manifold in the subspace (as a function of position on the viewsphere)
- recognition: nearest neighbour assuming dense sampling of object pose variations in the training set.



A B C D




A B C D

# Summary

**The generative model:**

- PCA finds the subspace (of a specified dimension) that maximizes projected signal variance.
- A single Gaussian model is naturally associated with a PCA representation. The principal axes are the principal directions of the Gaussian's covariance.

**Issues:**

- The single Gaussian model is often rather crude. PCA coeff's can exhibit significantly more structure (cf. Murase & Nayar).
- As a result of this unmodelled structure, detectors based on single Gaussian models are often poor. See the Matlab tutorial `detectEigenEyes.m`.
- We discuss alternative detection strategies later in this course.

# Further Readings


Belhumeur et al (1997) Eigenfaces vs. fisherfaces: Recognition using class specific linear projection. *IEEE Trans. PAMI*, 19(7):711-720

Chennubhotla, C. and Jepson, A.D. (2001) Sparse PCA (S-PCA): Extracting multi-scale structure from data. *Proc. IEEE ICCV*, Vancouver, pp. 641-647.

T. Cootes, G. Edwards, and C.J. Taylor, Active Appearance Models, *Proc. IEEE ECCV*, 1998.

Golub, G. and van Loan, C. (1984) Matrix Computations. Johns Hopkins Press, Baltimore.

Murase, H. and Nayar, S. (1995) Visual learning and recognition of 3D objects from appearance. *Int. J. Computer Vision* 14:5–24.

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Moghaddam, B., Jebara, T. and Pentland, A. (2000) Bayesian face recognition. *Pattern Recognition*, 33(11):1771-1782

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P. Viola, and M.J. Jones, Robust real-time face detection, Int. J. of Computer Vision, 2004.