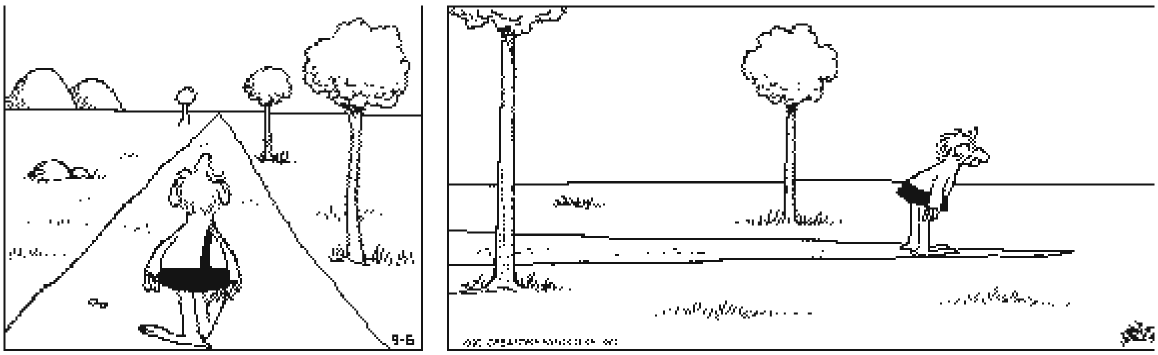
Parallel Lines Project to Intersecting Lines



As an application of (7), consider a set of parallel lines in 3D, say

$$\vec{X}_k^h(s) \;=\; \begin{pmatrix} \vec{X}_k^0 \\ 1 \end{pmatrix} + s \begin{pmatrix} \vec{t} \\ 0 \end{pmatrix}.$$

Here $\vec{X}_k^0$, for $k = 1, \ldots, K$, and $\vec{t}$ are 3D vectors in the world coordinate frame. Here $\vec{t}$ is the common 3D tangent direction for all the lines, and $\vec{X}_k^0$ is an arbitrary point on the k^{th} line.

Then, according to equation (7), the images of these points in homogeneous coordinates are given by

$$\vec{p}_k^{\,h}(s) \;=\; M\vec{X}_k^h(s) \;=\; \vec{p}_k^{\,h}(0) + s\vec{p}_t^{\,h},$$

where $M = M_{int}M_{ext}$ is a 3×4 matrix, $\vec{p}_t^{\,h} = M(\vec{t}^{\,T}, 0)^T$ and $\vec{p}_k^{\,h}(0) = M((\vec{X}_k^0)^T, 1)^T$. Note $\vec{p}_t^{\,h}$ and $\vec{p}_k^{\,h}(0)$ are both constant vectors, independent of s. Converting to standard pixel coordinates, we have

$$\vec{p}_k(s) \;=\; \frac{1}{\alpha(s)}\vec{p}_k^{\,h}(0) + \frac{s}{\alpha(s)}\vec{p}_t^{\,h},$$

where $\alpha(s) = p_{k,3}^h(s)$ is third component of $\vec{p}_k^{\,h}(s)$. Therefore we have shown $\vec{p}_k(s)$ is in the subspace spanned by two constant 3D vectors. It is also in the image plane, $p_{k,3} = 1$. Therefore it is in the intersection of these two planes, which is a line in the image. That is, lines in 3D are imaged as lines in 2D. (Although, in practice, some lenses introduce “radial distortion”, which causes the image of a 3D line to be bent. However, this distortion can be removed with careful calibration.)

In addition it follows that $\alpha(s) = p_{k,3}^h(0) + \beta s$ where $\beta = p_{t,3}^h = (0, 0, 1)M(\vec{t}^{\,T}, 0)^T$. Assuming $\beta \neq 0$, we have $1/\alpha(s) \to 0$ and $s/\alpha(s) \to 1/\beta$ as $s \to \infty$. Therefore the image points $\vec{p}_k(s) \to (1/\beta)\vec{p}_t^{\,h}$, which is a constant image point dependent only on the tangent direction of the 3D lines. This shows that the images of the parallel 3D lines $\vec{X}_k^h(s)$ all intersect at the image point $(1/\beta)\vec{p}_t^{\,h}$.