CSC 238H
Midterm, Spring 2003

St. George Campus

Duration — 50 minutes

Student Number: |_|_|_|_|_|_|_|_|_|

Last Name: ____________________

First Name: ____________________

No aids allowed.

*Do **not** turn this page until you have received the signal to start.*

1: ______/10

2: ______/10

3: ______/10

TOTAL: ______/30

Good Luck! PLEASE HAND IN

Question 1. [10 MARKS]

For each of the statements below, indicate whether it is true or false by circling the corresponding word. *You do NOT need to justify your answers.*

a. $15n^2 + 10n \in \mathcal{O}(n^3)$ — $\boxed{\text{TRUE}}$ / FALSE

b. $5n^2 \log(n) + 10n \in \Omega(n^2)$ — $\boxed{\text{TRUE}}$ / FALSE

c. $n \log(n) + n \in \Theta(n)$ — TRUE / $\boxed{\text{FALSE}}$

d. $\frac{1}{500}n - 10\sqrt{n} \in \Omega(n)$ — $\boxed{\text{TRUE}}$ / FALSE

e. $50n^2 \log(n) + 30n \in o(n^3)$ — $\boxed{\text{TRUE}}$ / FALSE

f. $50n^2 \log(n) - 30n \in o(51n^2 \log(n))$ — TRUE / $\boxed{\text{FALSE}}$

g. $15n^2 - 10n \in \Theta(20n^2)$ — $\boxed{\text{TRUE}}$ / FALSE

h. $3\log^2(n) - \log(n) \in \mathcal{O}(\log(n))$ — TRUE / $\boxed{\text{FALSE}}$

i. $\log(\sqrt{n}) \in \mathcal{O}(\sqrt{\log(n)})$ — TRUE / $\boxed{\text{FALSE}}$

j. $\log(n^2 \log(n)) \in \mathcal{O}(\log(n))$ — $\boxed{\text{TRUE}}$ / FALSE

Question 2. [10 MARKS]

Consider the function $f(n)$ defined by

$$\begin{aligned} f(0) &= 3, \\ f(1) &= 0, \\ f(n) &= 2f(n-2) + n^2, \quad \text{for } n \geq 2. \end{aligned} \tag{1}$$

For example, $f(2) = 10$, $f(3) = 9$, and $f(4) = 36$. Prove that $f(n+1) > f(n)$ for all integers $n \geq 3$.

Proof. Let $S(n)$ be the statement "$f(n+1) > f(n)$". We will use complete induction to prove that $S(n)$ is true for all $n \geq 3$.

Base Case. Equation 1 implies that $f(2) = 10$, $f(3) = 9$, $f(4) = 36$, and $f(5) = 43$. Therefore $f(4) > f(3)$ and $f(5) > f(4)$. And so $S(3)$ and $S(4)$ are true.

Let $k \geq 5$ be an arbitrary integer.

Induction Hypothesis. Suppose $S(j)$ is true for $3 \leq j < k$.

Induction Step. We need to prove $S(k)$ is true. Since $k \geq 5$ we have $k > k - 2 \geq 3$ and, by the induction hypothesis, $S(k-2)$ must be true. Therefore $f(k-1) > f(k-2)$. We use this fact below.

Using equation (1) we find,

$$\begin{aligned} f(k+1) &= 2f(k+1-2) + (k+1)^2, \quad \text{by equation (1), since } k+1 \geq 2, \\ &> 2f(k-1) + k^2, \quad \text{since } (k+1)^2 > k^2, \\ &> 2f(k-2) + k^2, \quad \text{since we showed above that } f(k-1) > f(k-2), \\ &= f(k), \quad \text{by equation (1), since } k \geq 2. \end{aligned}$$

Therefore $f(k+1) > f(k)$ and so $S(k)$ is true.

By mathematical induction it follows that $S(n)$ is true for all $n \geq 3$.

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Question 3. [10 MARKS]

Use an appropriate loop invariant to prove that the following program is correct.

IntLog(x, b)
Precondition: x, b are integers such that $x \geq 1$, $b \geq 2$.
Postcondition: Returns the integer k such that $b^k \leq x < b^{k+1}$.

```
1   k := 0
2   n := 1
3   while n * b <= x do
4       k := k + 1
5       n := n * b
6   end while
7   return k
```

Proof. Define $L(i)$ to be the loop invariant: "If exactly i iterations have been completed, then $k_i = i$ and $n_i = b^i \leq x$." We will use induction to prove that $L(i)$ is true for all $i \in \mathbb{N}$.

Base Case: By lines 1 and 2 of the program, we find $k_0 = 0$ and $n_0 = 1$. Since $b > 0$ we have $b^0 = 1$. By the precondition $x \geq 1$, so we have $n_0 = b^0 = 1 \leq x$. Hence $L(0)$ is true.

Let i be an arbitary natural number.

Induction Hypothesis. Suppose $L(i)$ is true.

Induction Step. We need to prove $L(i+1)$ is true. If the loop exits before iteration $(i+1)$, then $L(i+1)$ is trivially true. Otherwise, when the loop body begins execution for the $(i+1)^{st}$ time then, from the induction hypothesis $L(i)$, we have $k_i = i$ and $n_i = b^i \leq x$. Moreover, the loop condition must be satisfied to allow iteration $i+1$, so $n_i b \leq x$. By lines 4 and 5 we then have $k_{i+1} = k_i + 1 = i + 1$ and $n_{i+1} = n_i b = b^{i+1}$. Finally, since we know from the loop condition that $n_i b \leq x$, we find that $n_{i+1} = n_i b \leq x$. This proves that $L(i+1)$ is true.

Therefore we conclude that $L(i)$ is true for all $i \in \mathbb{N}$.

Partial Correctness. If the program terminates immediately after iteration i then, from the loop invariant $L(i)$, we have $k_i = i$ and $n_i = b^i \leq x$. Moreover, the loop condition $n_i b \leq x$ must fail to hold, and therefore $n_i b > x$. It follows that $n_i = b^{k_i} \leq x < b^{k_i+1}$ Therefore $k = k_i$ satisfies the postcondition upon termination.

Termination. Consider the sequence $< s_0, s_1, \ldots >$ where s_i is defined only if the loop completes at least i iterations, in which case $s_i = x - n_i$. By the precondition and the loop invariant $L(i)$, each of x, b, and $n_i = b^i$ are integers with $n_i \leq x$. Therefore $s_i = x - n_i \geq 0$ is a natural number. Moreover, if s_i and s_{i+1} exist, then

$$\begin{aligned} s_{i+1} &= x - n_{i+1} = x - n_i b, \text{ by the loop invariant } L(i+1), \\ &< x - n_i, \text{ since } 1 < b \text{ and } 0 < n_i = b^i \text{ together imply } n_i < n_i b, \\ &= s_i. \end{aligned}$$

Therefore $< s_0, s_1, \ldots >$ is a decreasing sequence of natural numbers. Therefore it must be a finite sequence. Therefore the program must terminate.

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